GEOMETROGRAPHY WITH APPLICATIONS TO THE INSTRUMENTS OF THE DRAFTSMAN

THESIS

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MASTER OF ARTS

By

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PREFACE

This preface takes the form of an acknowledgment.. The subject of this thesis was suggested by Dr. H. J. Ettlinger; and during its preparation he has been generous with his valuable time, his kindly criticisms, and his encouragement. The writer, who has been guided to a deeper understanding of the subject, wishes to express his sincere appreciation to Dr. Ettlinger for the invaluable assistance he has given.

CONTENTS

CHAPTER I

INTRODUCTION

The purpose of this thesis is to devise elementary constructions which are the bases of all construction; first, with compasses alone; second, with compasses plus the ruler; third, with compasses and ruler plus the triangles. Along with these constructions a system of Geometrography will be used to measure each construction and a comparison will be made of the three methods used. The central idea is to investigate and ascertain whether a number of the elementary constructions given in our Plane Geometries can be made practically as short and in some cases shorter by use of the compasses alone; and further to see how much shorter they can be made by introducing the triangles in Plane Geometry. Suppose we have a construction problem which can be solved by compasses alone and also with ruler and compasses; and further, suppose "compasses-alone" construction is somewhat longer than the ruler and compasses construction. Which method is more desirable for accuracy? When a line is drawn by the aid of a straight edge, it may be in error for three reasons: first, the straight edge may not coincide with the point or points through which the line is to be drawn; second, the straight edge is likely to be moved as the line is being ruled; and third, the angle of elevation of the pen may not remain constant as the line is being ruled. When points are located by use of the compasses, they may be in error for only one reason: the operation of making the points of the compasses coincide with the reference points. However, this error is not so likely with points of the compasses as with ruler. There is no chance of error from the points of the compasses being moved as the circle is drawn for one leg of the compasses is anchored in the paper. Further, there is no chance of error from the angle of elevation of the compasses with the plane of the paper not being constant. Thus from the point of view of accuracy a "compass-alone" construction is more desirable than a ruler-and-compasses construction, even though the "compasses-alone" construction is longer. The errors that may occur by use of the triangles are the same as those occurring by the use of the ruler.

CHAPTER II

GEOMETROGRAPHY

As time passes, problems of construction and theorems in Geometry are being constructed and proved. many different ways. For example, we know five simple methods, all different, to construct the parallel postulate. The Pythagorean Theorem has been proved in many ways. The problem then arises as to which method of construction of the parallel postulate is the most accurate; also, the problem as to which proof of the Pythagorean Theorem is the shortest and simplest. We are now living in a period of economy and we are always on the look-out for short methods. We might give a discussion of short methods of proof for different theorems; but leaving that aside, we will develop in this chapter a discussion of the ways of measuring different constructions.

In order to make a comparison of the brevity, simplicity, and accuracy of two or more constructions, we must assign values to each operation in the steps of the constructions. Therein arises a difficulty. If one could devise a method of grading a construction so we might compare it with other constructions as to the characteristics of simplicity and accuracy, he would make a definite advance in this field of mathematics.

Several attempts have been made in this direction. J. Steiner (1796-1863) in the year 1833 suggested the problem of investigating and obtaining some method of measuring geometric exercises to determine the simplest, shortest, and most exact construction of the same exercise. A practical system by which a construction could be measured was first introduced by C. Weiner (1826-96) in his textbook Lehrbuche der Dorstellenden Geometrie (Leipzig, Teubner, 1884). The question was first critically investigated by Lemoine in 1888 in his work, Geometrographie, ou l'Art des Constructions Geometriques (Paris, 1902). This subject was also investigated by J. Roensch in his work, Planimetrische Konstruktionen in Geometriographischer Ansfuhrung (Leipzig, Teubner, 1904). The system developed by Lemoine was revised by Nach Bernes, and at present seems to be the best system in use.

The system of Geometrography used by Lemoine is this:

Operation	Description
R_1	Make the straight edge pass through one given point.
R_2	Rule a straight line.
C_1	Make one compass leg coincide with a given point.
C_2	Make one compass leg coincide with any point on a given line.
C_3	Describe a circle.

Thus if these operations occur respectively L_1, L_2, m_1, m_2, m_3 times in a construction, its symbol is

$$L_1R_1 + L_2R_2 + m_1C_1 + m_2C_2 + m_2C_3;$$

the sum of the coefficients $L_1 + L_2 + m_1 + m_2 + m_3$, which is the total number of operations, is the coefficient of simplicity, and the total number of coincidences $L_1 + m_1 + m_2$ is the coefficient of exactitude. The difference between these coefficients gives $L_2 + m_3$, which is the number of lines or circles drawn.

The system of Geometrography used by Bernes was as follows:

S --operation of drawing any straight line.

S_1--operation of drawing any straight line through a given point.

S_2--operation of drawing a straight line through two given points.

γ--operation of drawing any circle.

γ_1--operation of drawing any circle with given center.

γ_2--operation of drawing a circle with given center through a given point.

γ_3--operation of drawing a circle with given radius and given center.

Now if in some construction a straight line is drawn $\underline{l}$ times, a line through a given point $\underline{l_1}$ times,...., and a circle with given radius and given center m_3 times, its symbol would be this:

$$1\delta + l_1\delta_1 + l_2\delta_2 + m\gamma + m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3.$$

The coefficient of simplicity would be:

$$1 + 2l_1 + 3l_2 + m + 2m_1 + 3m_2 + 4m_3.$$

The coefficient of exactitude, which is the number of coincidences, is this:

$$l_1 + 2l_2 + m_1 + 2m_2 + 3m_3.$$

The number of operations is:

$$1 + l_1 + l_2 + m + m_1 + m_2 + m_3.$$

Example: Given three points not in a straight line, to find the center of the circle which passes through these points. In this construction we draw three circles with given centers and two straight lines through two given points. According to Lemoine its symbol would be $3C_1 + 3C_3 + 4R_1 + 2R_2$, coefficient of simplicity 12, coefficient of exactitude 7, and number of lines and circles drawn 5. According to Bernes its symbol would be $3\gamma_1 + 2\delta_2$, coefficient of simplicity 12, coefficient of exactitude 7, and number of operations 5. Thus we see the two systems give same results, but the system of Bernes is more easily applied.

The system used in this thesis will be closely related to that of Bernes and the following symbols will be used:

L_0--operation of ruling any straight line.

L_1--operation of ruling any straight line through a given point.

L_2--operation of ruling a straight line through two given points.

C_0--operation of describing any circle.

C_1--operation of describing any circle with given center; or operation of describing any circle through a given point.

C_2--operation of describing a circle with given center through a given point.

C_3--operation of describing a circle with given radius and given center.

T_2--operation of erecting a perpendicular to a given line at a given point in the line with triangles.

T_3--operation of dropping a perpendicular from a given external point to a given line; or the operation of constructing the parallel postulate with triangles.

C. S.--coefficient of simplicity, which is the number of coincidences plus the number of lines and circles drawn and which is calculated by multiplying the coefficient of the letters by their subscripts increased by one and taking the sum.

C. E.--coefficient of exactitude, which is the number of coincidences and which is obtained by multiplying the coefficient of the letters by their subscripts and taking the sum.

N. O.--number of operations which is the number of lines and circles drawn and is obtained by the sum of coefficients or C. S. - C. E.

CHAPTER III

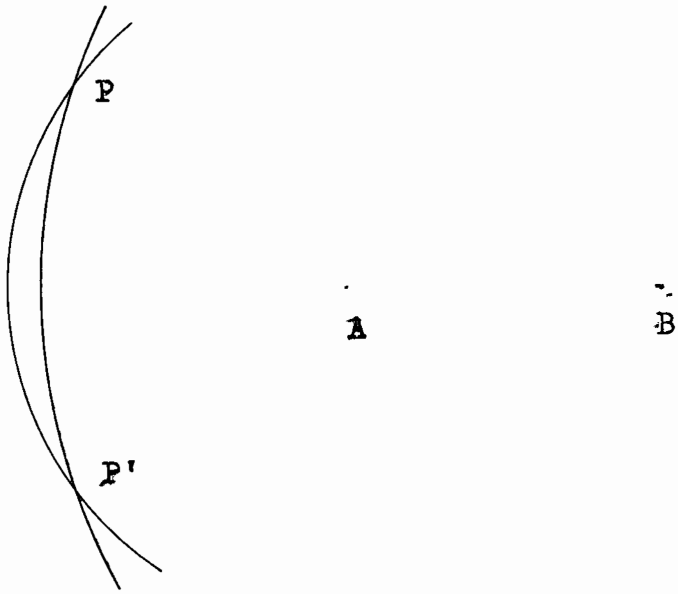
"COMPASSES-ALONE" CONSTRUCTIONS

In making a construction we always have given certain lines or points which we call reference lines or points and from these we locate other lines and points until we have our required construction made. For a line to be given we must always have two points on the line given, for two points determine only one line. When we say the line AB is given, we mean the two points A and B are given and these two points fix a straight line through them. In this chapter we are given points and with the compasses we locate other points and if in a problem a line is to be found, if we can locate two points of the line, it will be determined.

The method of writing up the construction part of each problem is similar to the method used by H. P. Hudson in her book on Ruler and Compasses. The first column gives the points as they were located. Each point is the intersection of two arcs, two lines, or an arc and a line; and the two arcs, two lines, or the arc and the line that locate each point are put on the same line with this point but in column two or three--all repetitions are put in column three. Column four gives the geometrographical symbol for each point.

Construction I

To let fall a $\perp$ upon a given line from a given external point.



Given line AB and external point P, to find a line $\perp$ to AB from P.

Construction:

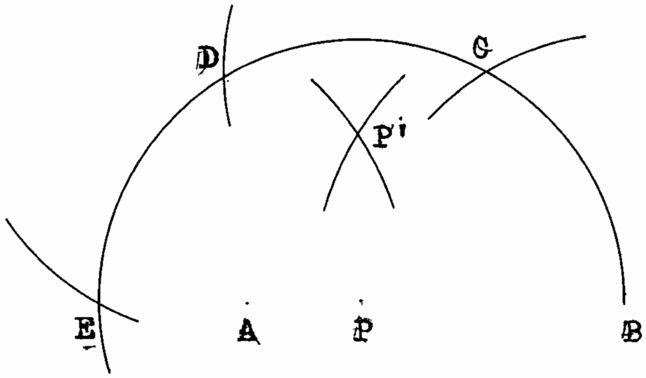
P'	A(P) B(P)		2 C_2

Symbol: 2 C_2; C. S. = 6; C. E. = 4; N. O. = 2.

Proof: "Two points each equidistant from the extemities of a given line determine the $\perp$ bisector."

Construction II

At a given point in a given line, to erect a $\perp$ to the given line.



Given the point P on the given line AB, to erect a $\perp$ to AB at P.

Method I.

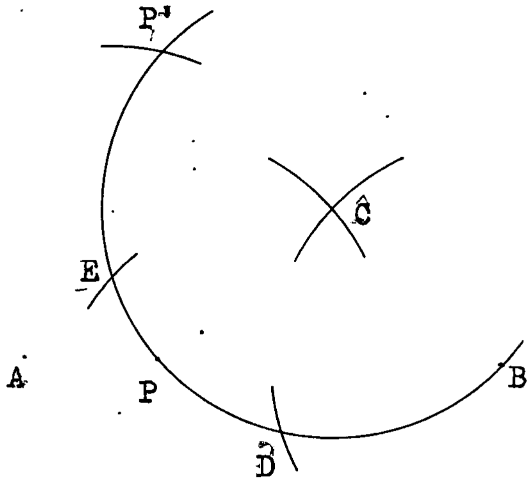
Construction:

C	P(B)B(PB)		C_2+C_1
D	C(PB)	P(B)	C_1
E	D(PB)	P(B)	C_1
P'	E(P')B(P')		2 C_1

Symbol: S C_1+C_2; C. S.=13; C. E.=7; N. O.=6.

Proof: "The radius will divide the sémi-circle into three equal parts." "Two points each equidistant from the extremities of a line determine the $\perp$ bisector."

Method II.



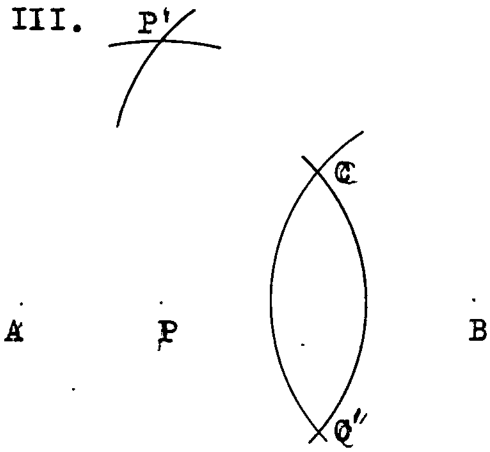
Construction:

C	P(C)B(C)		2 C_1
D	B(PC)C(PC)		2 C_1
E	D(PC)	C(PC)	C_1
P'	E(PC)	C(PC)	C_1

Symbol: 6 C_1; C. S.=12; C. E.=6; N. O.=6.

Proof: "An angle inscribed in a semicircle is a right angle."

Method III.



Construction:

C	P(C)B(C)		2 C_1
C'		P(C)B(C)	
P'	C(PC)P(CC')		C_1+C_2

Symbol: 3 C_1+C_2; C. S.=9; C. E.=5; N. O.=4.

Proof: Quadrilateral PC'CP' is a parallelogram and PB is $\perp$ to CC' $\therefore$ it is $\perp$ to PP'.

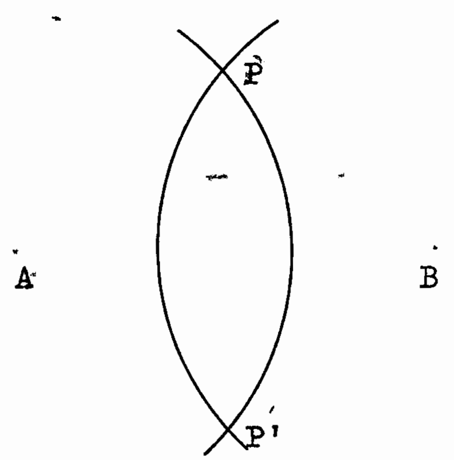
Corallary: To produce a line segment its own length.

Construction:

Apply method I and we get the symbol 3 C_1+C_2 which gives us a C. S.=9, C. E.=5, and N. O.=4.

Construction III

To find the perpendicular bisector of a given line segment.



Given the line segment AB,

to find the perpendicular bisector.

Construction:

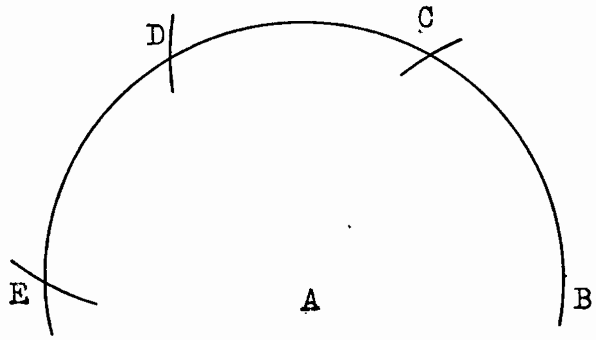
P	A(P) B(P)		Z C_1
P'		A(P) B(P)	

Symbol: 2 C_1; C. S.=4; C. E.=2; N. O.=2.

Proof: "Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.

Construction IV

To construct a right triangle with hypotenuse twice a given side (constructing a line = $\sqrt{3}$)



Given line segment AB, to construct a right triangle with AB one side and hypotenuse twice AB.

Method: I.

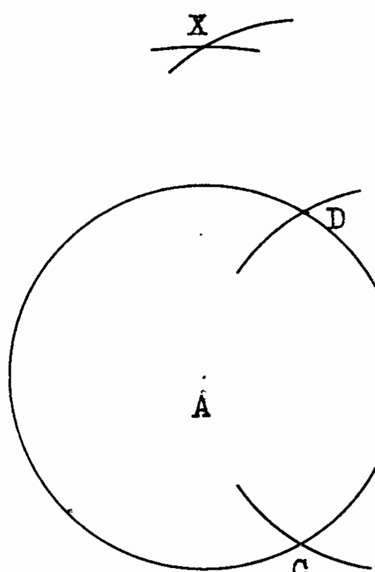
Construction:

C	A(B) B(A)		C_1+C_2
D	C(AB)	A(B)	C_1
E	D(AB)	A(B)	C_1

Symbol: 3 C_1+C_2; C. S.=9; C. E.=5; N. O.=4.

Proof: "An angle inscribed in a semicircle is a right angle."

Method II.



Construction:

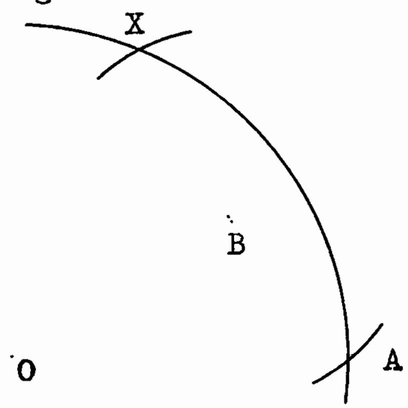
C	A(B) B(A)		C_1+C_2
D		A(B) B(A)	
X	D(B) A(CD)		C_1+C_3

Symbol: $2\ C_1 + C_2 + C_3$; C. S. = 11; C. E. = 7; N. O. = 4.

Proof: Angle BAX is rt. angle by Construction II--Method III. Angle BAD and ADB are 60° (equilateral triangle). Angles DAX and DXA are = (opposite = sides), but $\angle$ DAX = 30°. Then $\angle$ XDA = 120°. Therefore $\angle$ XDB is straight angle and DB = XD by construction.

Construction V

To double an angle.



Given the angle AOB,

required to double angle AOB

Construction:

X	O(A) B(A)		2 C_2

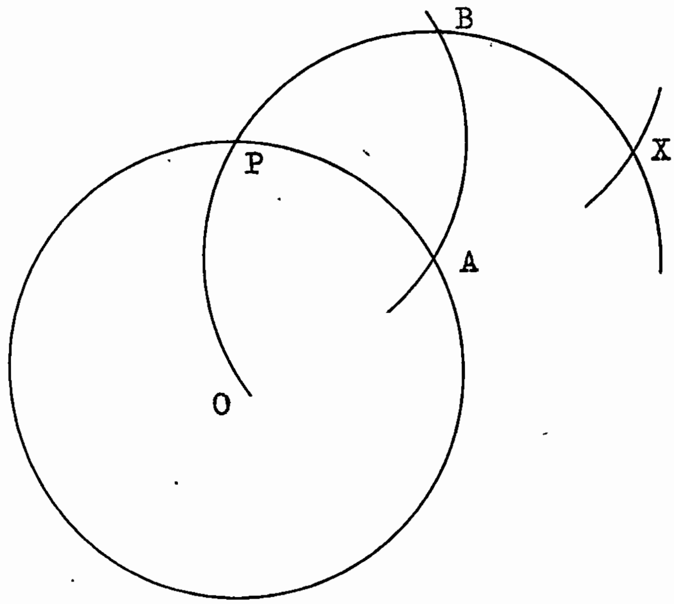
Symbol: 2 C_2; C. S. = 6; C. E. = 4; N. O. = 2.

Proof: The two triangles AOB and BOX have three sides of one equal respectively to three sides of the other. Therefore angle BOX is equal to angle

AOB and consequently angle AOX is double angle AOB.

Construction VI

To draw a tangent to a circle at a point on the circle.



Given circle with center O and point P on the circle, to draw a tangent to the circle at P.

Method I.

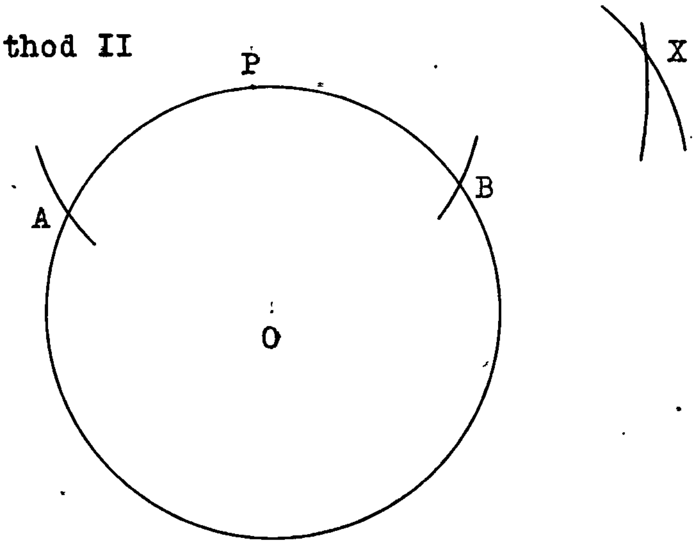
Construction:

A	P(O)	O(P)	C_2
B	A(O)	P(O)	C_1
X	B(OP)		

Symbol: $2\ C_1 + C_2$; C. S. = 7; C. E. = 4; N. O. = 3.

Proof: "An angle inscribed in a semicircle is a right angle'. "A line $\perp$ to a radius at its extremity on circle is tangent to the circle."

Method II



Construction:

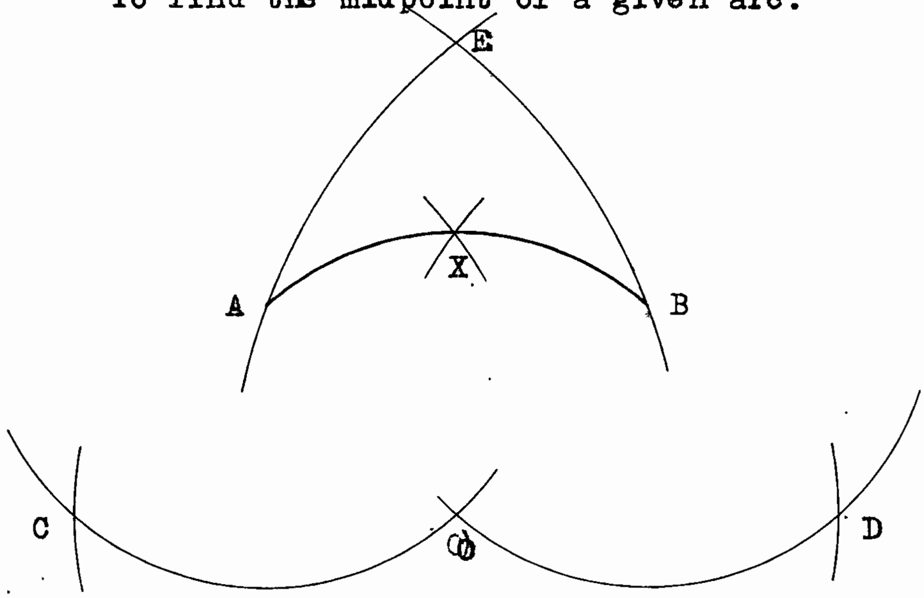
A	P(O)	O(P)	C_2
B		P(O) O(P)	
X	B(PO) P(AB)		$C_1 + C_3$

Symbol: $C_1 + C_2 + C_3$; C. S. = 9; C. E. = 6; N. O. = 3.

Proof: XP is perpendicular to OP at P by Construction II--Method III.

Construction VII

To find the midpoint of a given arc.[1]



Given arc AB the arc of a circle with center O, to find the midpoint of arc AB

Construction:

C	A(O) O(AB)		$C_2 + C_3$
D	B(O)	O(AB)	C_1
E	D(A) C(B)		$C_1 + C_2$
X	C(OE)	AB	C_3

Symbol: $2\ C_1 + 2\ C_2 + 2\ C_3$; C. S. = 18; C. E. = 12; N. O. = 6.

[1]Mascheroni, Lorenzo: La Geometria del Compasso, Pavia, 1797, pp. 32-35.

Loria, Gino: Vorlesungen Uber Dorstellende Geometrie, Leipzig and Berlin, 1907, pp. 4, 5.

Proof:

Angle AOB = 2a

$AB = 2\ \overline{OA} \cdot \sin a$

$\angle\ BOC = 90^{\circ} + a$

Then in triangle BCO,

$$\overline{BC}^2 = \overline{CO}^2 + \overline{OB}^2 + 2\ \overline{OB} \cdot \overline{OC} \sin a$$
$$= \overline{OA}^2 + \overline{AB}^2 + 2\ \overline{OA} \cdot \overline{AB} \sin a$$
$$= \overline{OA}^2 + 2\ \overline{AB}^2 = \overline{OA}^2 + 2\ \overline{OC}^2$$
$$= \overline{CE}^2 = \overline{OC}^2 + \overline{DE}^2$$

Then $\overline{DC}^2 + \overline{DE}^2 = \overline{OA}^2 + 2\ \overline{OC}^2$

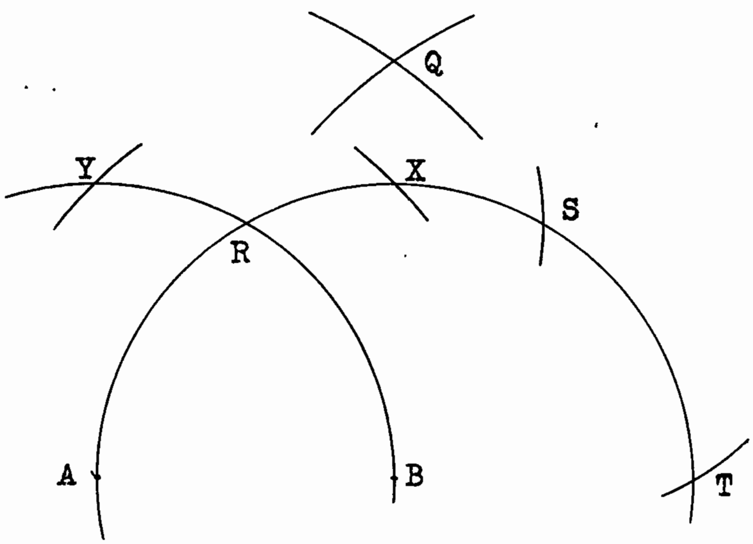
$$\overline{OE}^2 = \overline{OA}^2 + \overline{OC}^2 = \overline{CX}^2 = \overline{OC}^2 + \overline{OX}^2$$

$\overline{OA}^2 + \overline{OC}^2 = \overline{OC}^2 + \overline{DX}^2$

Therefore OA = OX.

Construction VIII

To construct a square given one side.



Given the line segment AB, to

Construct a square with side AB.

Construction:

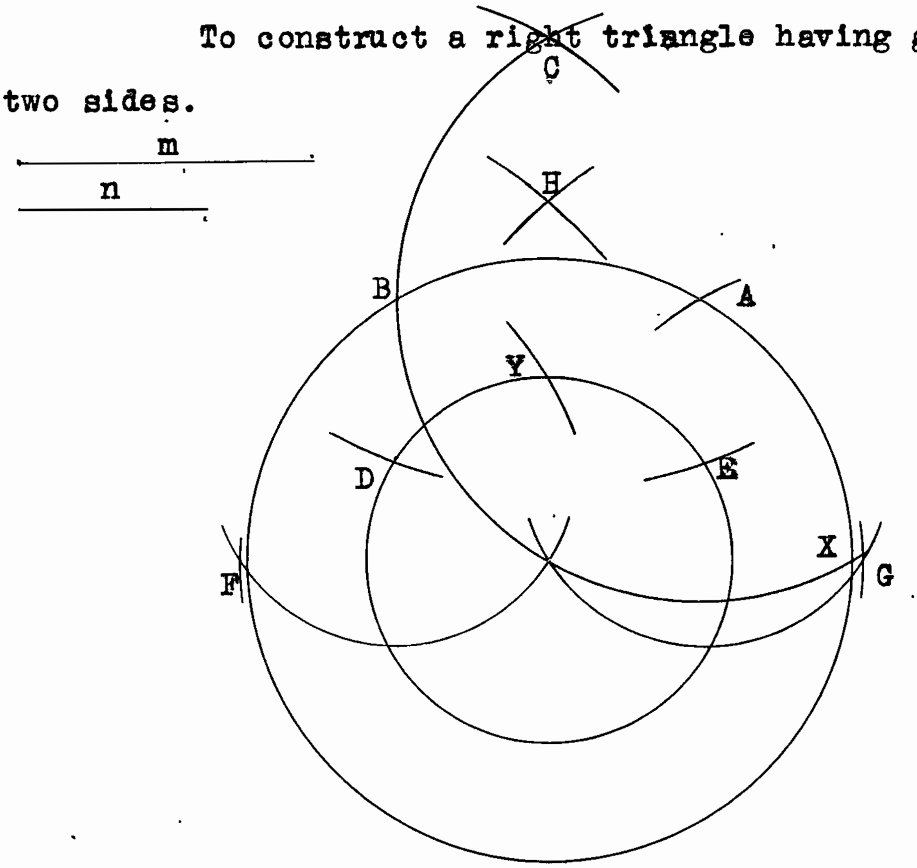
R	A(B) B(A)		$C_1 + C_2$
S	R(AB)	B(A)	C_1
T	S(AB)	B(A)	C_1
Q	A(S) T(AS)		$C_1 + C_2$
X	A(BQ)	B(A)	C_3
Y	B(Q)	A(B)	C_0

Symbol: $C_0 + 4 C_1 + 2 C_2 + C_3$; C. S. = 19; C. E. = 11; N. O. = 8.

Proof: X is midpoint of semicircle ARST (Construction VII). $\triangle$s BAY and ABX are congruent (corresponding sides are =); thus $\angle$ BAY is rt. $\angle$. Therefore AY and XB are $\parallel$ ($\perp$ to same line) and quadrilateral ABXY is a parallelogram and since two adjacent sides are = and one $\angle$ is a rt. $\angle$, the figure is a square.

Construction IX

To construct a right triangle having given the two sides.



Given line segments m and n, to construct a right triangle with these segments as sides.

Construction:

A	O(m) X(m)		$C_1 + C_2$
B	A(m)	O(m)	C_1
C	B(m)	A(m)	C_1
D	C(D) O(n)		$C_1 + C_3$

E		C(D) O(n)	
F	D(O) O(DE)		$C_2 + C_3$
G	E(O)	O(DE)	C_1
H	F(E) G(D)		$C_1 + C_2$
X			
Y	G(OH)	O(n)	C_3

Symbol: $6\ C_1 + 3\ C_2 + 3\ C_3$; C. S. = 33; C. E. = 21; N. O. = 12.

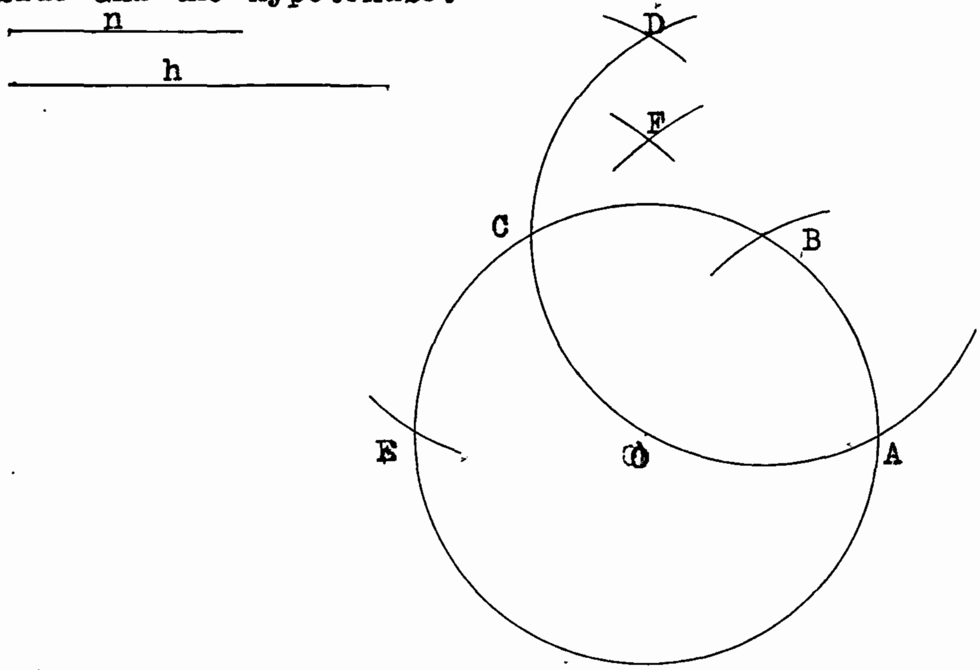
Proof: CO is perpendicular to OX by Construction II. Y is the intersection of line CO and the circle O with radius n. X was taken a point on the circle o with radius m. Therefore triangle XOY is a right triangle and its sides are m and n.

Corallary. To construct a square equivalent to two given squares.

Construction: This construction will be a repetition of Construction IX and a repetition of construction VIII which will give a C. S. = 52, C. E. = 32, and N. O. = 20.

Construction X

To construct a right triangle having given one side and the hypotenuse.



Given the line segments h and n, to construct a right triangle with n one side and h the hypotenuse.

Construction:

A	O(n)		C_2
B	A(n)	O(n)	C_1
C	B(n)	O(n)	C_1
D	C(n)	B(n)	C_1
E	D(A)	On	C_2
F	A(h) E(h)		$C_1 + C_3$

Symbol: $4\ C_1 + 2\ C_2 + C_3$; C. S. = 18; C. E. = 11; N. O. = 7.

Proof: DO is perpendicular to AO by Construction II. Triangles DOA and DOE are congruent (having corresponding sides equal). Thus angle FOE is right angle and it follows that AOE is a straight line. Then if H is taken as a radius and E and A as centers and arcs described intersect at F, triangle AOF will be the required right triangle.

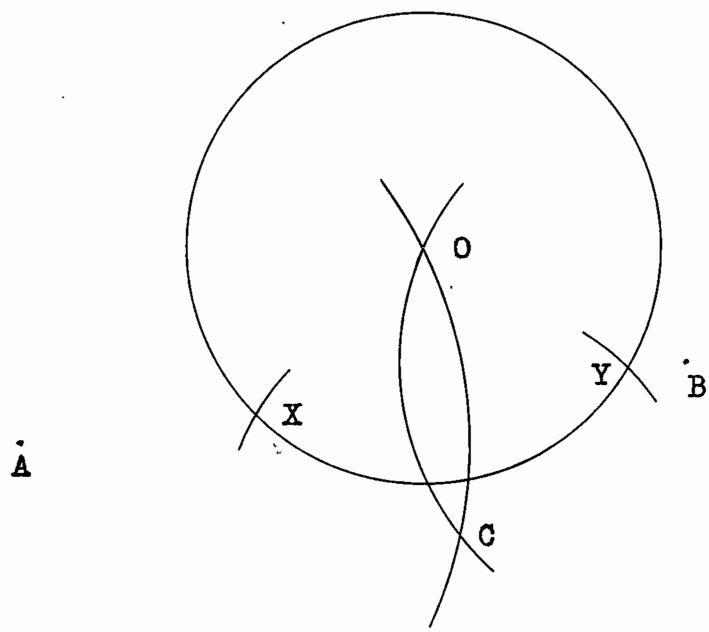
Corollary. To construct a square equivalent to the difference of two given squares.

Construction: This can be done by the repetition of the above construction and a repetition of Construction VIII which will give a C. S. = 37, C. E. = 22, and N. O. = 15.

Construction XI

To find the intersection of a given line and a given circle.

Case I. When the center of the circle is not on the given line.



Given circle with center O and the points A and B, to find the points where the line AB cuts the circle.

Construction:

C	B(O) A(O)		2 C_2
X, Y	C(O)	O	C_2

Symbol: 3 C_2; C. S. = 9; C. E. = 6; N. O. = 3.

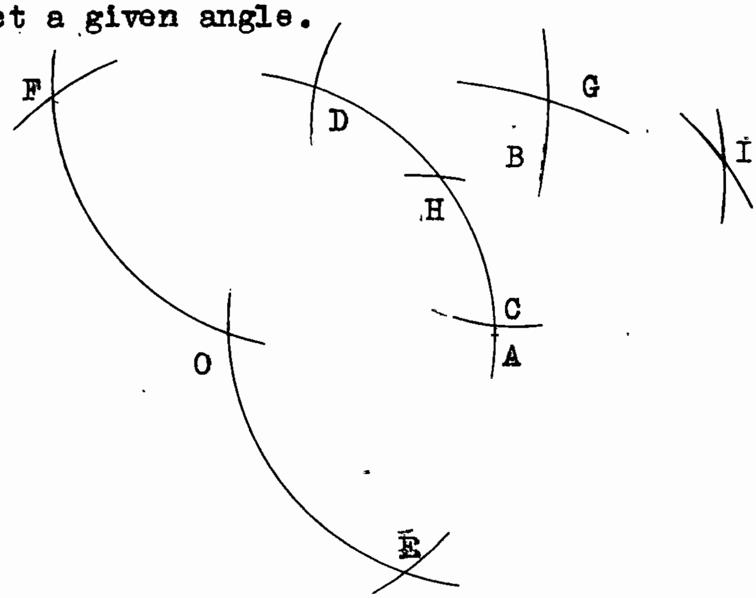
Proof: AB is the perpendicular bisector of OC (two points each equidistant from the extremities of a line determine the perpendicular bisector). Since all points equidistant from the extremities of a given line lie on the perpendicular bisector, then X and Y are on the line AB. This construction will work regardless of the position of A and B.

Case II. If O lies on line AB, then by Construction VII, the points of intersection may be found.

Symbol: 3 C_1 + 2 C_2 + 2 C_3; C. S. = 20; C. E. = 13; N. O. = 7.

Construction XII

To bisect a given angle.



Given the angle AOB, to

find the bisector of angle AOB.

Construction:

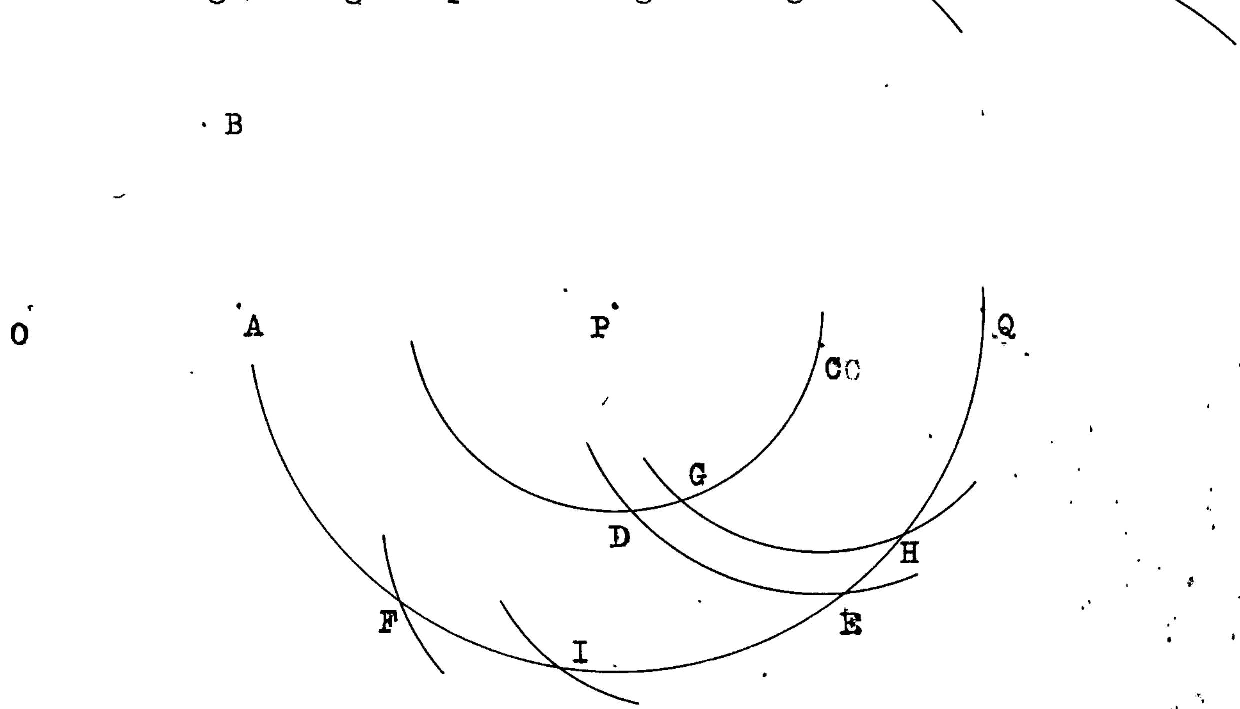
C	O(A) B(C)		$C_1 + C_2$
D		O(A) B(C)	
E	C(O) O(CD)		$C_2 + C_3$
F	D(O)	O(CD)	C_1
G	E(D) F(C)		$C_1 + C_2$
H	E(OG)	O(A)	C_3
I	H(I) A(I)		$2\ C_1$

Symbol: $5\ C_1 + 3\ C_2 + 2\ C_3$; C. S. = 27; C. E. = 17; N. O. = 10.

Proof: H is the intersection of OB and circle with center O by Construction XI--Case II. I is on bisector of angle AOB for $\triangle$ AOI and HOI have sides respectively equal.

Construction XIII

From a given point in a given line, to draw a line making an angle equal to a given angle.



Given the angle AOB and line PQ,

to construct an agnle upon PQ at P equal to angle AOB.

Method I.

Construction:

D	P(OA) C(OB)		2 C_3.
E	P(Q)	C(OB)	C_2
F	D(OB)	P(Q)	C_1

G	C(AB)	P(OA)	C_3
H		C(AB) P(Q)	
I	G(AB)	P(Q)	C_1
J	P(FE) Q(HI)		2 C_3

Symbol: $2\ C_1 + C_2 + 5\ C_3$; C. S. = 27; C. E. = 19; N. O. = 8.

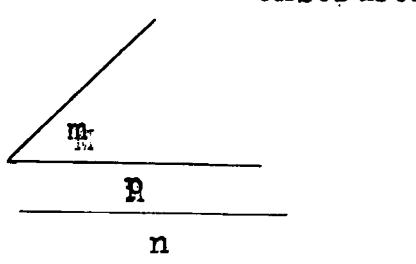
Proof: By Construction XIV PJ is the fourth proportional to OA, PQ, and OB; also QJ is the fourth proportional to OA, PQ, and AB. Therefore the triangles AOB and QPJ are similar and it follows that angle QPJ is equal AOB.

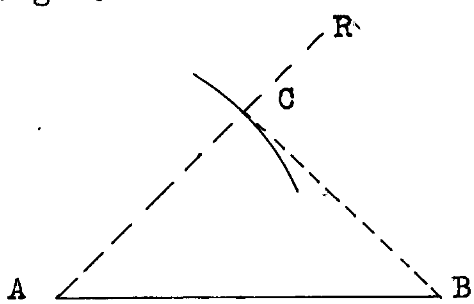
Method II.

By use of construction XI--case II, the intersection of PQ and circle with center P and radius OA may be found, call this intersection R. Draw R(AB) and P(OB) and call point of intersection T. The angle TPR is required angle for $\triangle$s TPR and AOB are congruent. C. of S. is 28, C. of E. is 19, and N. of O. is 9.

<u>Corollary 1</u>. To construct a triangle given the two sides and included angle.

Construction:

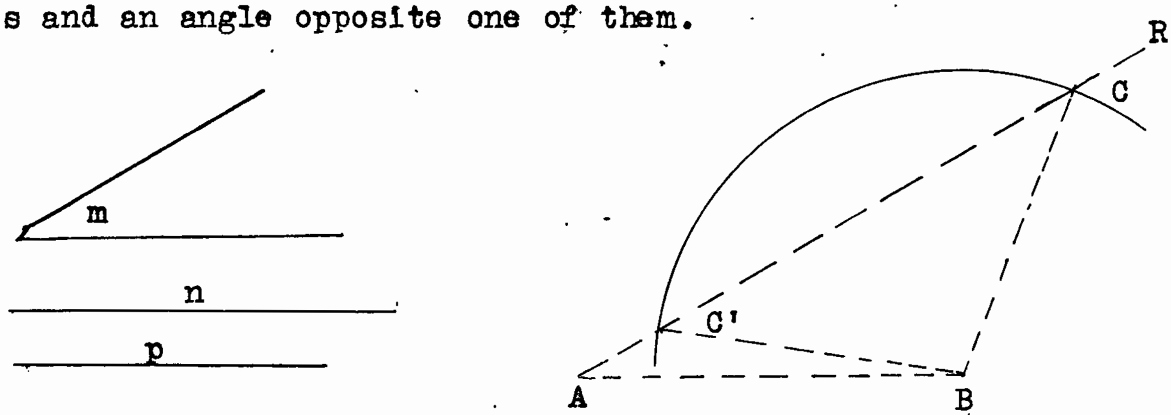




Given the line segments n and p and the angle m, to construct a triangle with n and p sides and angle m the included angle.

Construction: Upon AB, which was made equal to n, construct an angle equal to m, using construction XV method I, at A. Draw A(p) and use construction XI case II to find the intersection of A(p) and AR. Then ABC is required triangle. C. S. = 59; C. E. = 37; N. O. = 17.

Corollary 2. To construct a triangle given two sides and an angle opposite one of them.

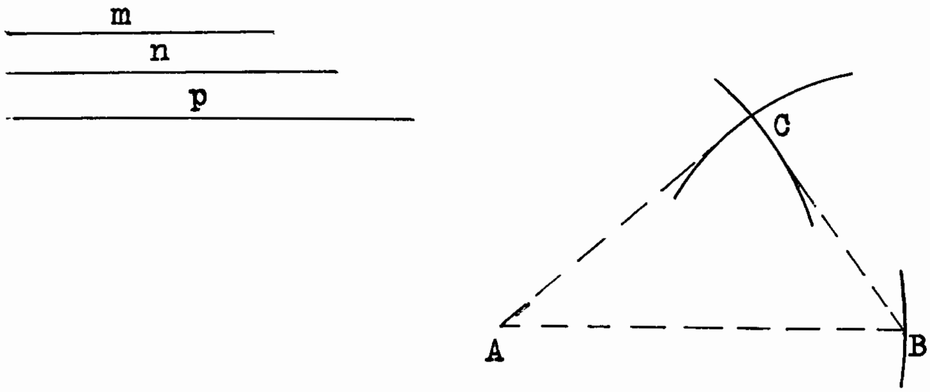


Given the two line segments n and p and angle m, to construct a triangle with angle m opposite one of the sides n or p.

Construction: Upon AB, which was made equal to n, construct an angle equal to m at A, using construction XV. Then draw B(p) cutting AR at C and C'. Now either ABC' or ABC is the required triangle. C.S. = 34; C. E. = 24; N. O. = 10.

Construction XIV

To construct a triangle given three sides.



Given three line segments m, n, and p,

to construct a triangle with these lines as sides.

Construction.

A	B(p)		C_2
C	B(n) A(m)		2 C_3

Symbol: $C_2 + 2\ C_3$; C. S. = 11; C. E. = 8; N. O. = 3.

Construction XV

Through a given external point to draw a line parallel to a given line.

Given the line AB and the external point C,

to draw a line through C parallel to AB.

Construction:

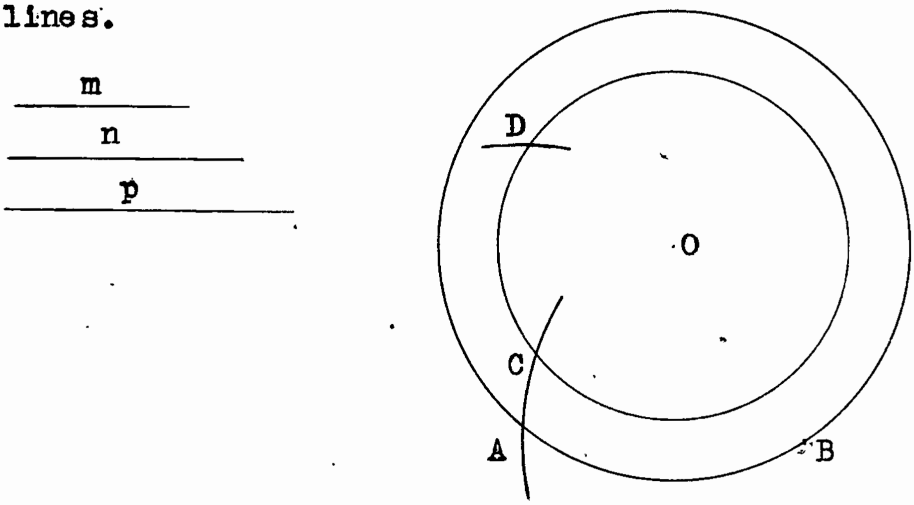
C	C(AB) B(AC)		2 C_3

Symbol: 2 C_3; C. S. = 8; C. E. = 6; N. O. = 8.

Proof: The quadrilateral ABDC is a parallelogram for its opposite sides are equal.

Construction XVI

To find the fourth proportional to three given lines.



Given three line segments m, n, and p, to find the fourth proportional to m, n, and p.

Construction:[2]

A	O(n) B(p)		$C_2 + C_3$
B			

[2] Mascheroni, Lorenzo: Geometria del Composso, pp. 68, 69. Loria, Gino: Vorlesungen Uber Dorstellende Geometrie, p. 5.

C	O(m)	B(p)	C_3
D	A(p)	O(m)	C_1

Symbol: $C_1 + C_2 + 2\ C_3$; C. S. = 13; C. E. = 9; N. O. = 4.

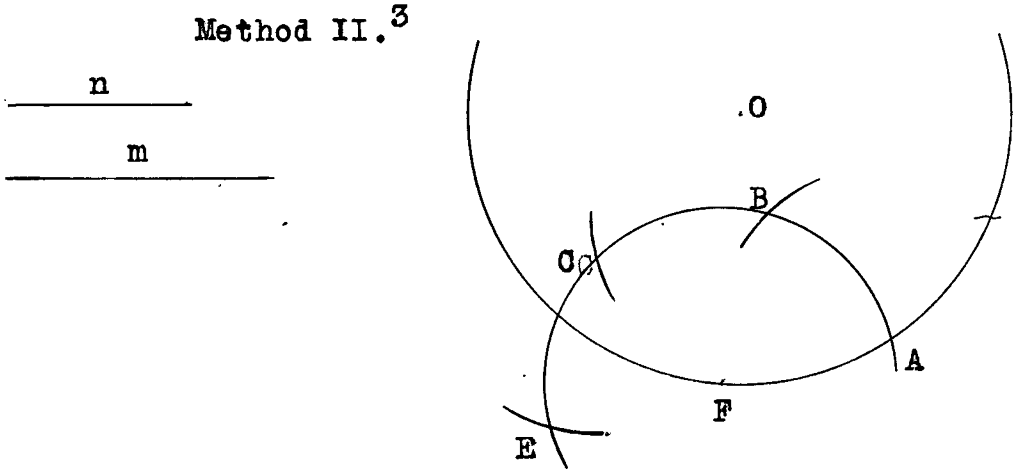
Proof: △s BCO and OAC are congruent. (Three sides respectively equal.) Then ∠ DOC = ∠ AOB (equals subtracted from equals). Therefore △s COD and AOB are similar (angle of one = angle of other and including sides proportional). Thus n:m = p:x.

<u>Corollary</u>. To find the third proportional to two given lines.

Method I.

Construction: This construction would be identical with the above construction except we would use <u>m</u> for <u>p</u>. The symbol would be the same, C. of S. is 13, C. of E. is 9, and N. of O. is 4.

Method II.[3]



[3] Mascheroni, Lorenzo: <u>Geometria del Composso</u>, p. 64.

Construction:

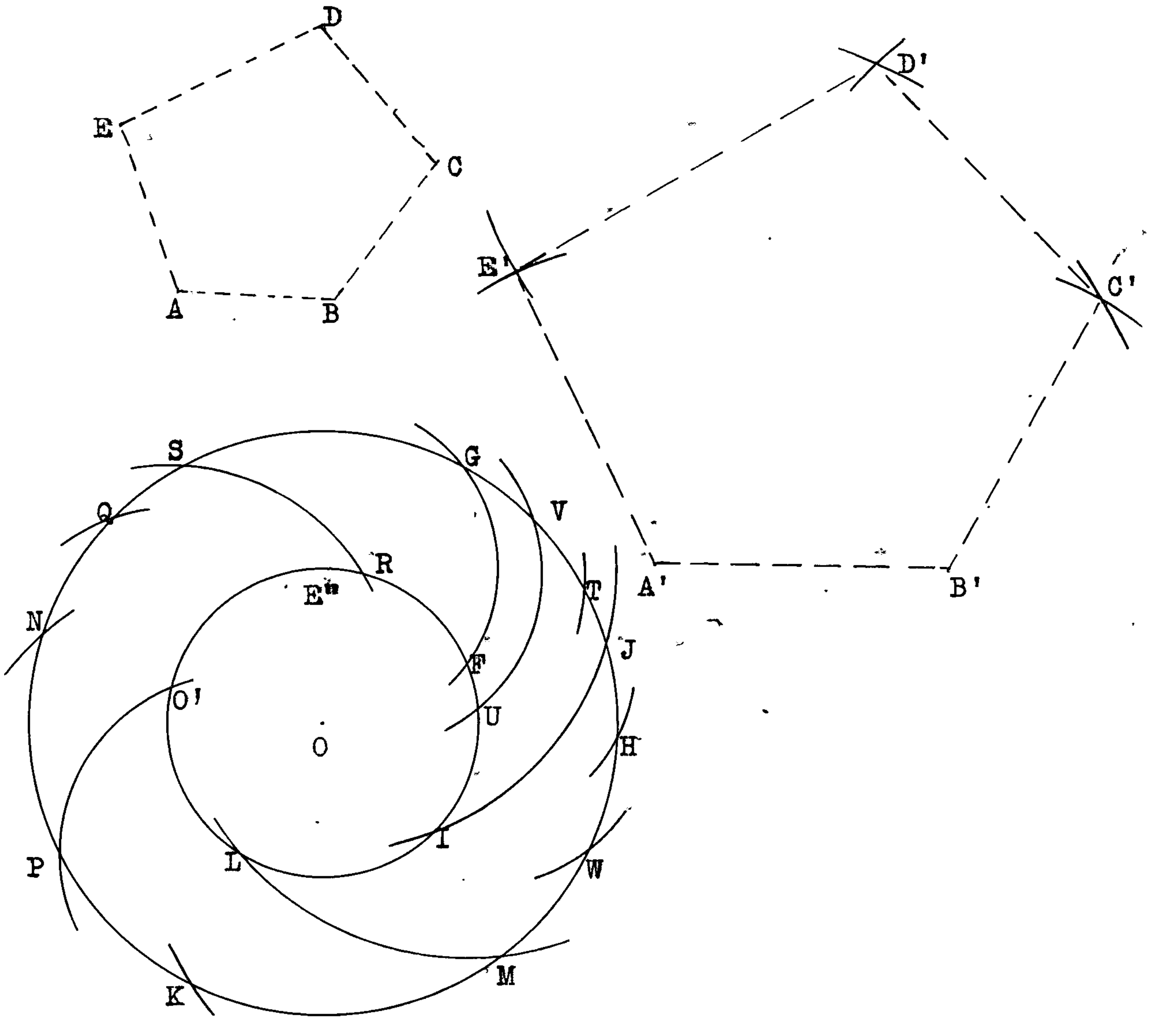
F	O(m)		C_2
A, D	F(n)	O(m)	C_3
B	A(n)	F(n)	C_1
C	B(n)	F(n)	C_1
E	C(n)	F(n)	C_1

Symbol: $3\ C_1 + C_2 + C_3$; C. S. = 13; C. E. = 8; N. O. = 5.

Proof: OF is $\perp$ to DA and DE is $\perp$ to DA. Therefore OF and DE are parallel and angle EDF = angle DFO. Thus the triangles DFE and DFO are similar and we have $\overline{DF}^2 = DE \cdot OF$. But DF = n and OF = m. Therefore $n^2 = m \cdot DE$ or $m:n = n:DE$.

Construction XVI

To construct a polygon similar to a given polygon.



Given the polygon ABCDE, to

construct a polygon similar to ABCDE upon the given line segment A'B' corresponding to AB.

Construction:

F	O(AB) E"(BC)		$C_2 + C_3$
G	O(A'B')	E"(BC)	C_3

H	F(BC)	O(A'B')	C_1
I	E"(AC)	O(AB)	C_3
J		E"(AC) O(A'B')	
K	I(AC)	O(A'B')	C_1
L	F(AD)	O(AB)	C_3
M		F(AD) O(A'B')	
N	L(AD)	O(A'B')	C_1
O'	L(CD)	O(AB)	C_3
P		L(CD) O(A'B')	
Q	O'(CD)	O(A'B')	C_1
R	O'(DE)	O(AB)	C_3
S		O'(CD) O(A'B')	
T	R(DE)	O(A'B')	C_1
U	R(AE)	O(AB)	C_3
V		R(AE) O(A'B')	
W	U(AE)	O(A'B')	C_1
C'	B'(GH) A'(JK)		2 C_3

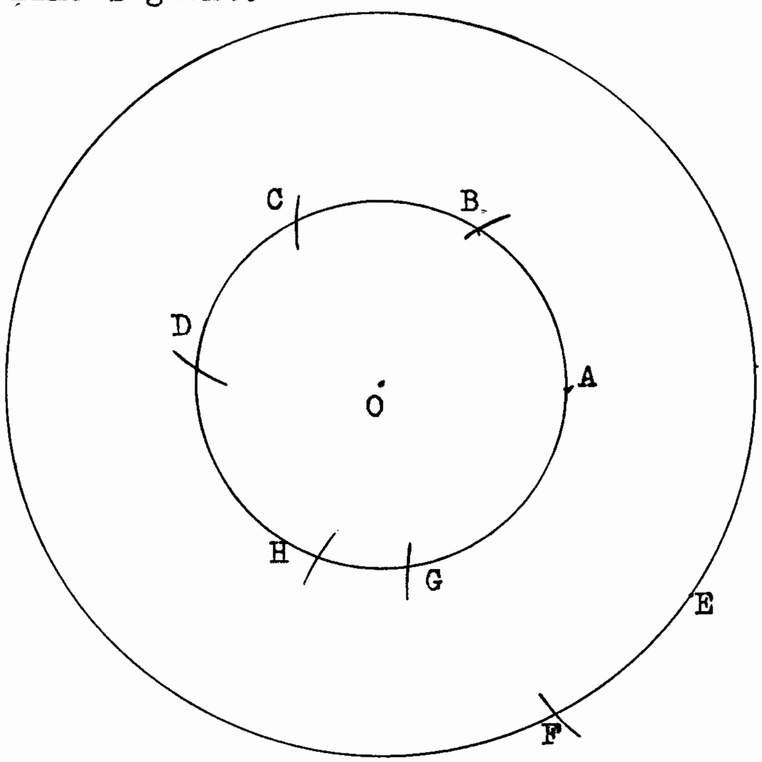
D'	A'(MN) C'(PQ)		2 C_3
E'	D'(ST) A'(VW)		2 C_3

Symbol: $6\ C_1 + C_2 + 13\ C_3$; C. S. = 67; C. E. = 47; N. O. = 20.

Proof: By construction XIV B'C' was found to be the fourth proportional to AB, A'B', and BC. Also AB : A'B' = AC : A'C', AB:: A'B' = AD : A'D', AB : A'B' = DC : D'C', AB:: A'B' = DC : D'C', AB : A'B' = AE : A'E'. Thus the polygons are similar having corresponding sides in proportion.

Construction XVIII

To find a line segment equal to one half of a given line segment.



Given line segment OA, to find a line segment equal to one half OA.

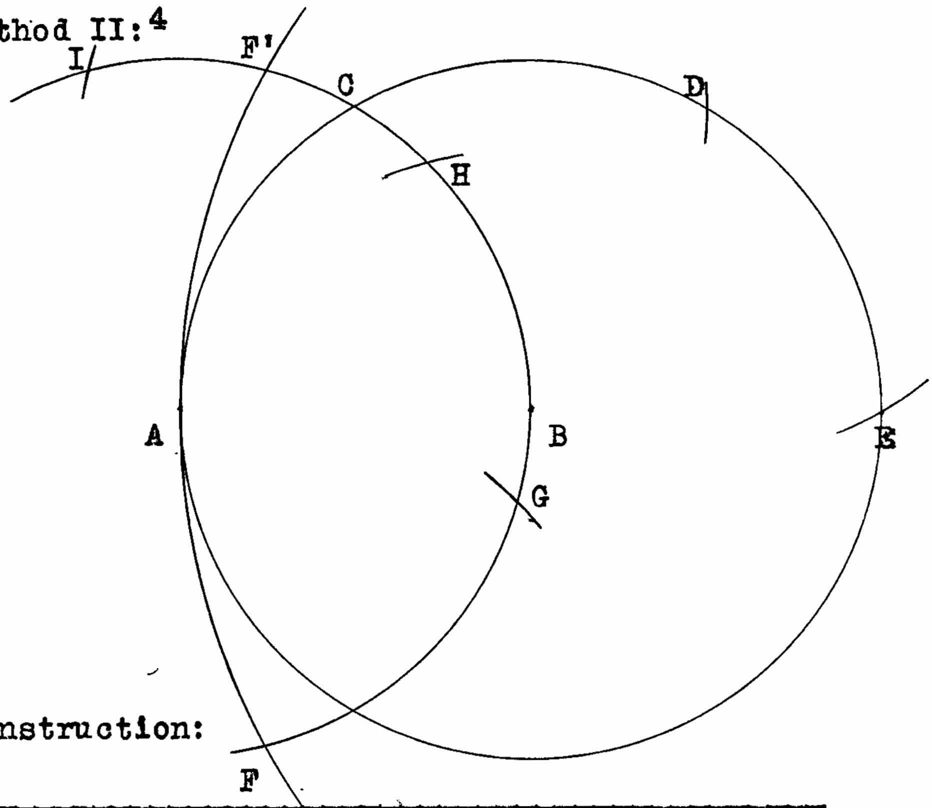
Construction:

B	O(A) A(OA)		$C_1 + C_2$
C	B(OA)	O(A)	C_1
D	C(OA)	O(A)	C_1
E			
F	E(OA) O(AD)		2 C_3
G		O(A) E(G)	
H	F(EG)	O(A)	C_1

Symbol: 4 $C_1 + C_2 + 2\ C_3$; C. S. = 19; C. E. = 12; N. O. = 7.

Proof: By Construction XIV AD : OA = OA : HG. By the corollary of Construction II AD is double OA, then OA is double HG.

Method II:[4]



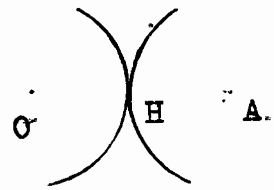
Construction:

C	A(B) B(A)		$C_1 + C_2$
D	C(AB)	B(A)	C_1
E	D(AB)	B(A)	C_1
F, F'	E(A)	A(B)	C_2
G	F(AB)	A(B)	C_3
H	G(AB)	A(B)	C_1
I	H(AB)	A B	C_1

4 Ibid., p. 38.

Symbol: $5 C_1 + 2 C_2 + C_3$; C. S. = 20; C. E. = 12; N. O. = 8.

Corollary. To find the midpoint of a line segment.



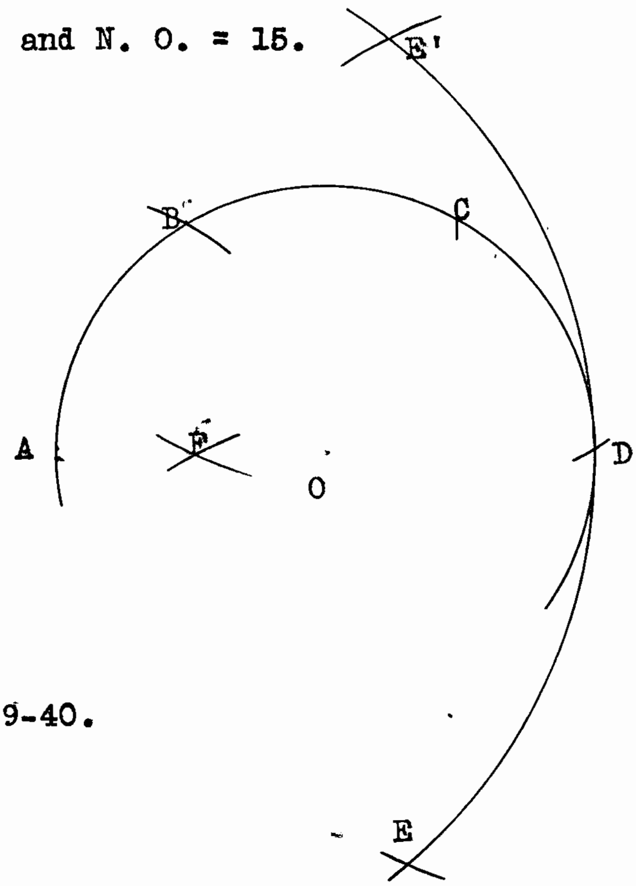
Method I.

Construction: Repeat the construction of the above construction, then draw A(GH) and O(GH) and the point of tangency will be the midpoint of OA. C. S. = 25; C. E. = 16; and N. O. = 9.

Method II.

A more accurate construction may be made by drawing A(GH) and by Construction XI Case II find the point of intersection of the circle and OA. This method gives C. S. = 43, C. E. = 28, and N. O. = 15.

Method III.[5]



5 Ibid., pp. 39-40.

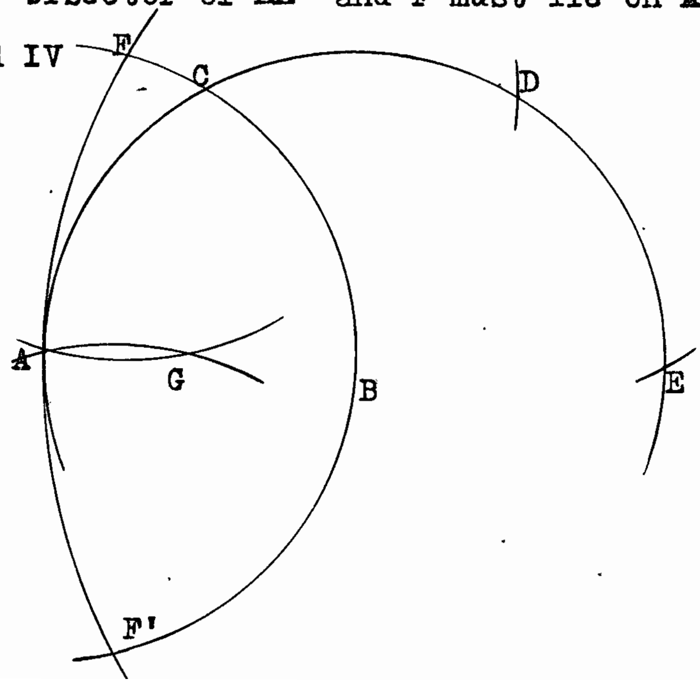
Construction:

B	O(A) A(O)		$C_1 + C_2$
C	B(OA)	O(A)	C_1
D	C(OA)	O(A)	C_1
E, E'	A(D) D(B)		$C_1 + C_2$
F	E(DB) E'(DB)		$2 C_1$

Symbol: $6 C_1 + 2 C_2$; C. S. = 18; C. E. = 10; N. O. = 8.

Proof: △s ADE' and FDE' are similar (being isosceles and having one side equal). Then $\frac{AD}{DE'} = \frac{DE'}{DF}$ or $\overline{DE'}^2 = AD \times DF$, but DE' = BD and $\overline{BD}^2 = 3\,\overline{AO}^2$. Thus $3\,\overline{AO}^2 = 2\ AO \times DF$; and 3 AO = 2 DF and AO = $\frac{2}{3}$ DF. Therefore OF = $\frac{1}{2}$ AO. Angle ADE' = angle FDE' for AD is perpendicular bisector of EE' and F must lie on AD.

Method IV



Construction:

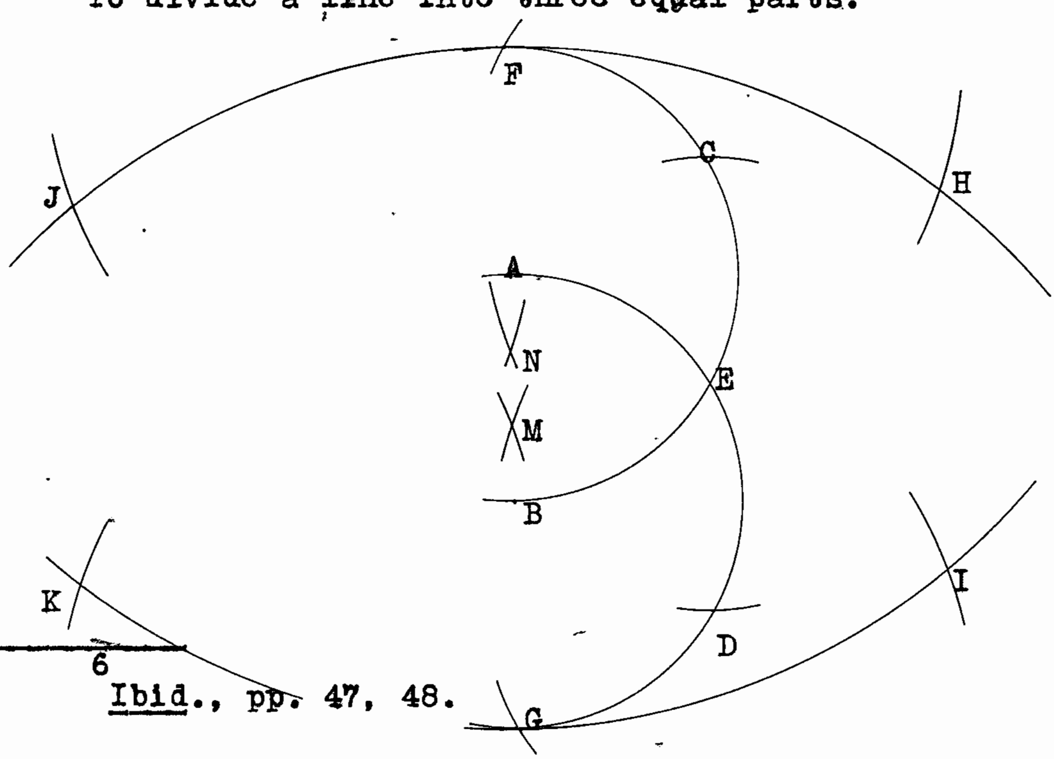
C	A(B) B(A)		$C_1 + C_2$
D	C(AB)	B(A)	C_1
E	D(AB)	B(A)	C_1
F, F'	E(A)	A(B)	C_2
G	F(A) F'(A)		$C_1 + C_2$

Symbol: $4\ C_1 + 3\ C_2$; C. S. = 17; C. E. = 10; N. O. = 7.

Proof: This proof is similar to the proof of method III.

Construction XIX

To divide a line into three equal parts.[6]



6 Ibid., pp. 47, 48.

Given the line segment AB, to divide AB into three equal parts.

Construction:

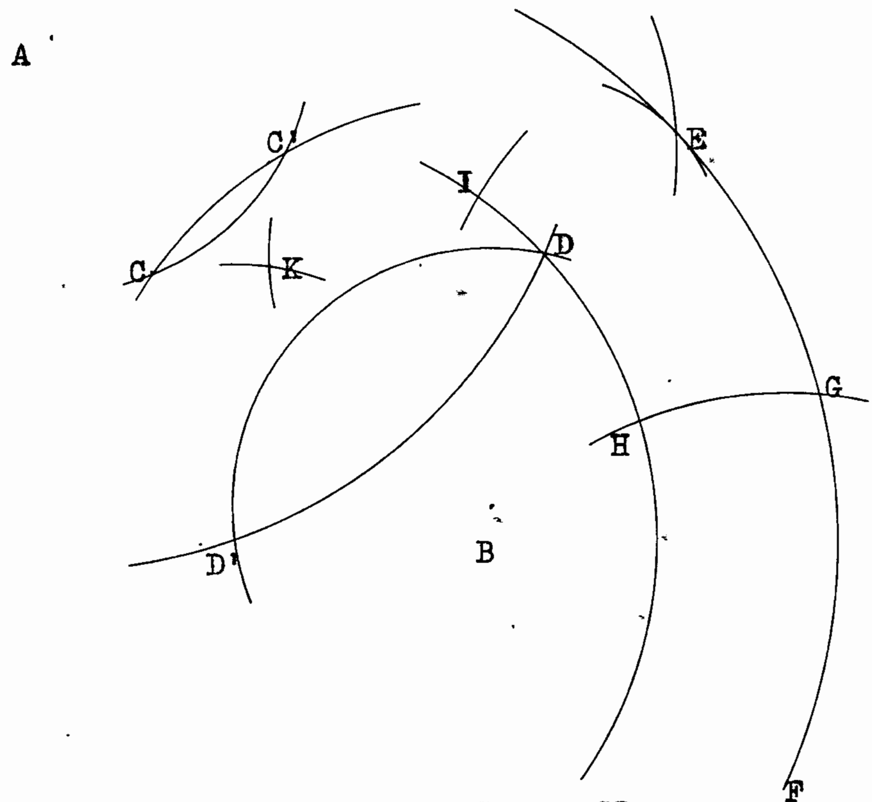
E	A(B) B(A)		$C_1 + C_2$
C	E(A)	A(B)	C_1
D		E(A) B(A)	
F	C(AB)	A(B)	C_1
G	D(AB)	B(A)	C_1
I, K	F(G) G(A)		2 C_2
J, H	G(F) F(B)		2 C_1
M	K(AG) I(AG)		2 C_1
N	J(AG) H(AG)		2 C_1

Symbol: $10\ C_1 + 3\ C_2$; C. S. = 29; C. E. = 16; N. O. = 13.

Proof: FG is the perpendicular bisector of IK and M and N are equidistant from I and K, therefore they must lie on FG. The triangles FGI and MGI are similar isosceles triangles. Then $\overline{IG}^2 = FG \times GM$ or $4\ \overline{AB}^2 = 3\ AB \cdot GM$; then $GM = \frac{4}{3} AB$ or $MB = \frac{1}{3} AB$.

Construction XX

To find the intersection of two given lines.[7]



Given the two lines AB and CD, to find their intersection.

Construction:

C'	A(C) B(C)		2 C_2
D'	A(D) B(D)		2 C_1
E	D(CC') C'(CD)		2 C_3

7 Ibid., p. 94.
Loria, Gino: Vorlesungen Uber Dorstellende Geometrie, p. 6.

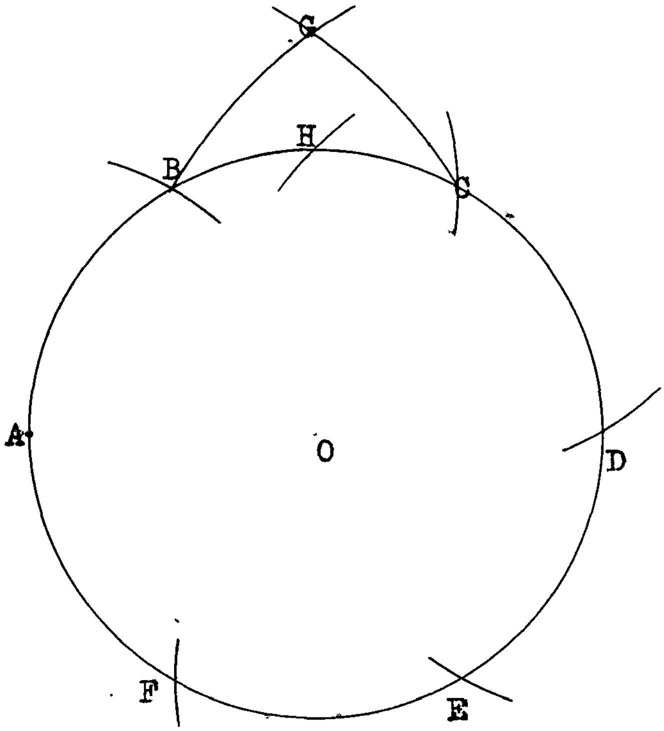
F			
G	F(D'C') D'(E)		$C_1 + C_3$
H	D'(D)	F(D'C')	C_1
I	G(D'C')	D'(D)	C_1
K	D'(HI) D(HI)		$C_1 + C_3$

Symbol: $6\ C_1 + 2\ C_2 + 4\ C_3$; C. S. = 34; C. E. = 22; N. O. = 12.

Proof: AB is the perpendicular bisector of the lines CC' and DD'; thus, D'DC'C is an isosceles trapezoid. CDEC' was constructed a parallelogram; and D'K was made the fourth proportional of D'E, D'D, and D'C'. Thus DK is parallel to C'E and DC is also parallel to C'E, therefore K is on CD. Since K is equidistant from D' and D it is also on AB. Therefore K is the intersection of AB and CD.

Construction XXI

To inscribe a regular hexagon in a circle.



Given circle with center O, to

inscribe a regular hexagon.

Construction:

B	A(OA)	O(A)	C_2
C	B(OA)	O(A)	C_1
D	C(OA)	O(A)	C_1
E	D(OA)	O(A)	C_1
F	E(OA)	O(A)	C_1

Symbol: $4\ C_1 + C_2$; C. S. = 11; D. E. = 6; N. O. = 5.

Proof: Triangle AOB is equilateral and therefore equiangular. Angle AOB = 60°, thus arc AB = 60° or 1/6 of a circle.

<u>Corollary</u>. To construct a line equal to the square root of two.

Construction:

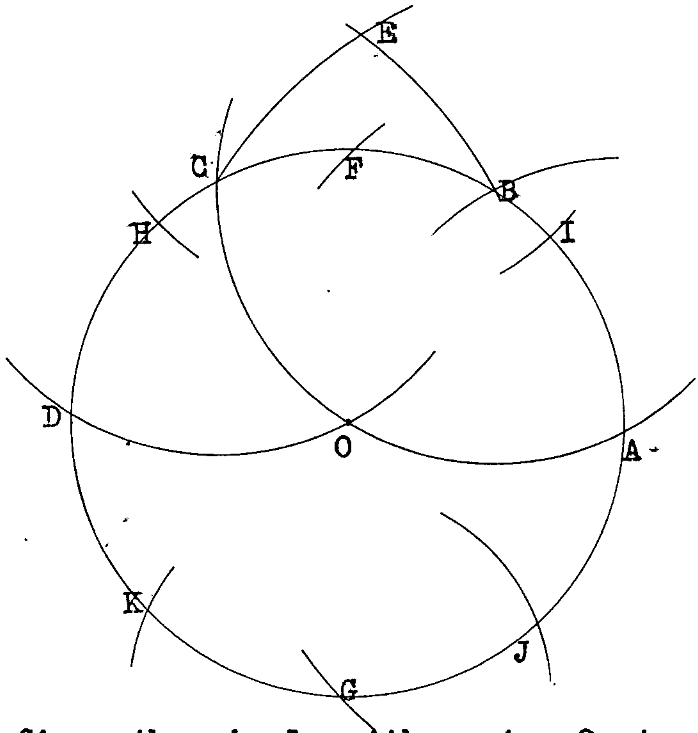
B	A(O) O(A)		$C_2 + C_1$
C	B(OA)	O(A)	C_1
D	C(OA)	O(A)	C_1
G	A(C) D(AC)	-	$C_1 + C_2$
O		O(A)	C_0
H	D(OG)	O(A)	

Symbol: $C_0 + 4\ C_1 + 2\ C_2$; C. S. = 15; C. E. = 8; N. O. = 7.

Proof: H is the midpoint of arc AHD (Construction VII). HOD is a right triangle with HO = 1 = OD. Thus, HD = $\sqrt{2}$ but HD was taken = OG, so OG = $\sqrt{2}$.

Construction XXII

To inscribe a square in a circle.



Given the circle with center O, to inscribe a square in the circle.

Construction:

B	A(O)	O(A)	C_1
C	B(O)	O(A)	C_1
D	C(O)		C_1
E	A(C) D(B)		$C_1 + C_2$
F	A(OE)	O A	C_3
G			

Symbol: $4\ C_1 + C_2 + C_3$; C. S. = 15; C. E. = 9; N. O. = 6.

Proof: AD is diameter (Construction II) and F is midpoint of arc ABCD (Construction VII). Therefore arcs AF, DF, AG, and DG are 90°.

<u>Corollary</u>. To inscribe a regular octagon in a circle.[8]

Construction: The construction will be the same as in the above construction plus the following:

H	E(OA) O(A)		C_3
I		E(OA) O(A)	
J	G(FH)	O(A)	C_3
K		G(FH) O(A)	

Symbol: $4\ C_1 + C_2 + 3\ C_3$; C. S. = 23; C. E. = 15; N. O. = 8.

Proof: AF = $\sqrt{2}$, then OE = $\sqrt{2}$ and OH = EH = 1. Therefore EHO is a right triangle and angle EOH is 45°. Thus arc DF is bisected and arc HF is equal one eight of circle.

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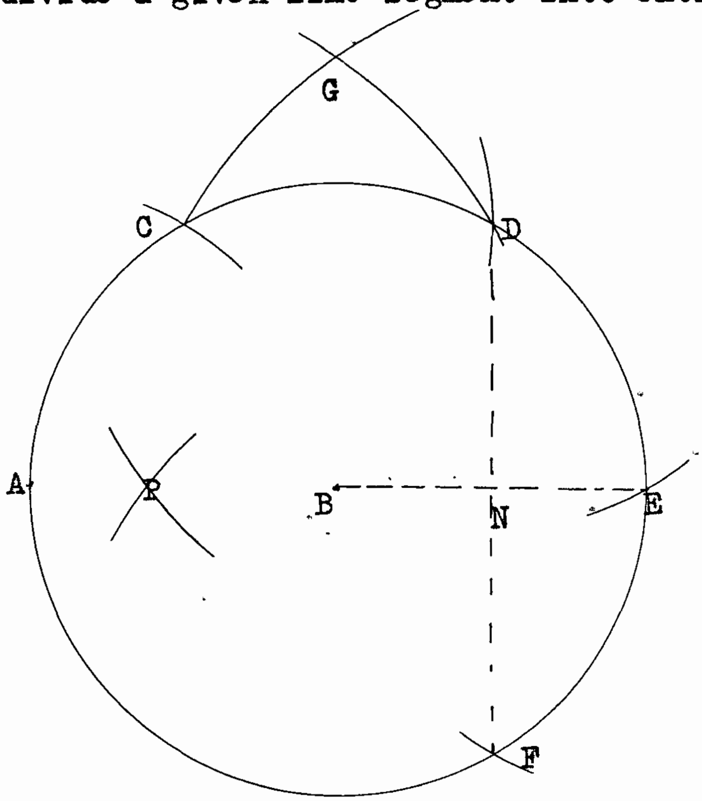
[8] Mascheroni, Lorenzo: <u>La Geometria del Composso</u>, pp. 23, 24.

Construction XXIII

To divide a given line segment into extreme and mean ratio.[9]



Given the line AB, to divide AB into extreme and mean ratio.

Construction:

C	B(A) A(B)		$C_1 + C_2$
D	C(AB)	B(A)	C_1
E	D(AB)	B(A)	C_1
F	E(AB)	B(A)	C_1

---

[9] Mascheroni, Lorenzo: La Geometria del Composso, p. 71

G	E(C) A(EC)		2 C_1
P	D(BG) F(BG)		$C_1 + C_3$

Symbol: 7 $C_1 + C_2 + C_3$; C. S. = 21; C. E. = 12; N. O. = 9.

Proof: $DP = CB = \sqrt{2}$ (Construction XXI, Cor. 2)

$DF = \sqrt{3}$ (Construction XXI, Cor. 3)

$DN = \frac{\sqrt{3}}{2}$

$\overline{PN}^2 = \overline{PD}^2 - \overline{ND}^2 = 2 - \frac{3}{4} = \frac{5}{4}$

$PN = \frac{1}{2}\sqrt{5}$

$PB = PN - BN = \frac{1}{2}\sqrt{5} - \frac{1}{2} = \frac{1}{2}(\sqrt{5} - 1)$

Let PB = X and if AB is divided into extreme and mean ratio,

$$1 : X = X : 1 - X$$

$$X^2 = -X + 1$$

$$X^2 + X - 1 = 0$$

$$X = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2} = \frac{1}{2}(\pm \sqrt{5} - 1)$$

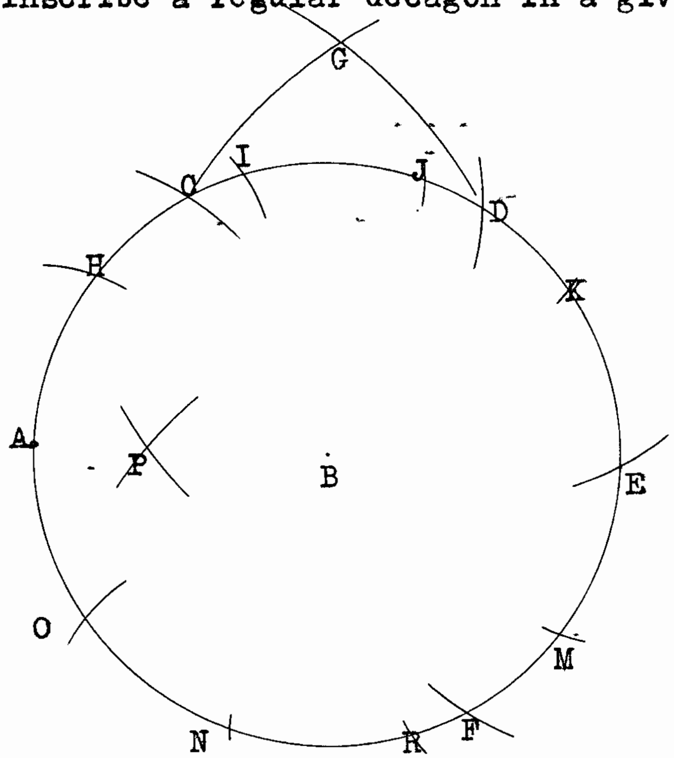
Now we have shown that

$$X = PB = \frac{1}{2}(\sqrt{5} - 1)$$

Therefore AB : PX = PX : AP. The negative sign is used when AB is divided externally.

Construction XXIV

To inscribe a regular decagon in a given circle.



Given the circle B(A), to

inscribe a regular decagon.

Construction:

C	A(B)	B(A)	C_2
D	C(BA)	B(A)	C_1
E	D(BA)	B(A)	C_1
F	E(BA)	B(A)	C_1
G	E(C) A(EC)		2 C_1
P	D(GB) F(GB)		C_1 + C_3

H	A(PB)	B(A)	C_3
I	H(PB)	"	C_1
J	I(PB)	"	C_1
K	J(PB)	"	C_1
L	K(PB)	"	C_1
M	L(PB)	"	C_1
R	M(PB)	"	C_1
N	R(PB)	"	C_1
O	N(PB)	"	C_1

Symbol: $14\ C_1 + C_2 + 2\ C_3$; C. S. = 39; C. E. = 22; N. O. = 17.

Proof: AB is divided into extreme and mean ratio at P (Construction XXIII). Triangles ABH and AHP are similar having an angle of one equal to an angle of the other and the including side in proportion. Since triangle ABH is isosceles, AHP is an isosceles. Now angle A = angle APH = angle AHB and PB = AH = HP. Thus angle PBH = angle PHB, Angle APH = 2 angle B. Then 5 $\angle$ B = 180°, and $\angle$ B = 36°, or arc AH = 36° or 1/10 of a circle.

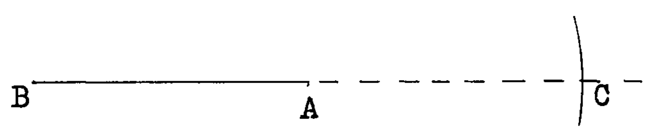
CHAPTER IV

RULER AND COMPASSES CONSTRUCTIONS

All the constructions made in Chapter III by means of compasses alone will be repeated in this chapter with ruler and compasses, with the exception of those which can not easily ~~wisely~~ be made. Examples are constructions I, II, III, XIV, and XXI. Where the proofs of the constructions are not given in this chapter, they will be found in the corresponding construction of Chapter III.

Corollary, Construction II

To produce a line segment its own length.



Given the line segment AB, to produce it its own length.

Construction:

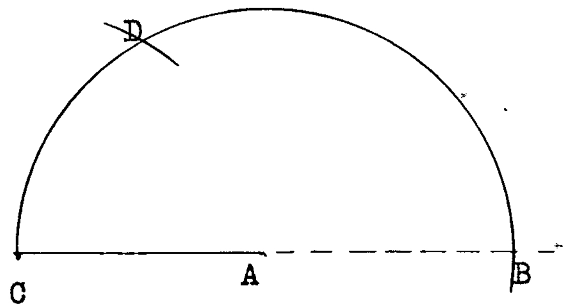
C	A(B) B.A		$L_2 + C_2$

Symbol: $L_2 + C_2$; C. S. = 6; C. E. = 4; N. O. = 2

Proof: The radius of a circle is one half the diameter.

Construction IV

To construct a right triangle with hypotenuse twice a given side (constructing a line = $\sqrt{3}$)



Construction:

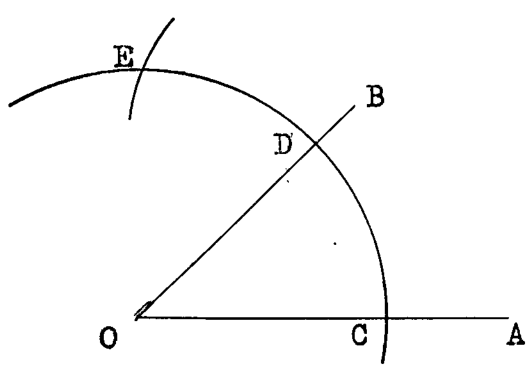
C	A(B) B A		$L_2 + C_2$
D	C(AB)	A(B)	C_1

Symbol: $C_1 + C_2 + L_2$; C. S. = 8; C. E. = 5; N. O. = 3

Proof: (An angle inscribed in a semicircle is a right angle.)

Construction V

To double an angle.



Given the angle AOB, to double it.

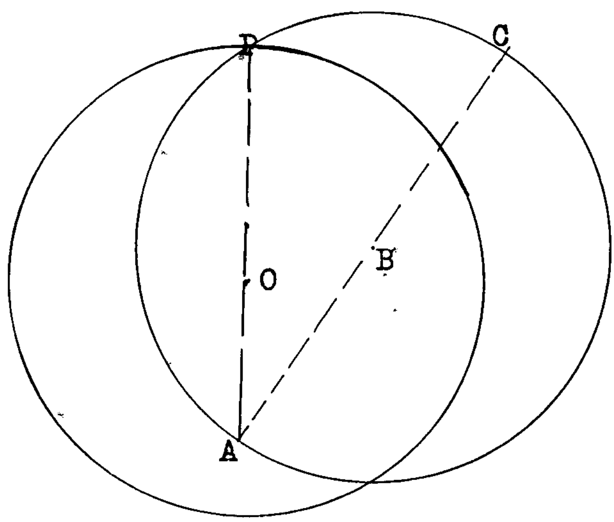
Construction:

C	O(C)	O A	C_1
D		O B O(C)	
E	D(C)	O(C)	C_2

Symbol: $C_1 + C_2$; C. S. = 5; C. E. = 3; N. O. = 2.

Construction VI

To draw a tangent to a circle at a point on the circle.



Given circle with center O, to draw a tangent to circle at P.

Construction:

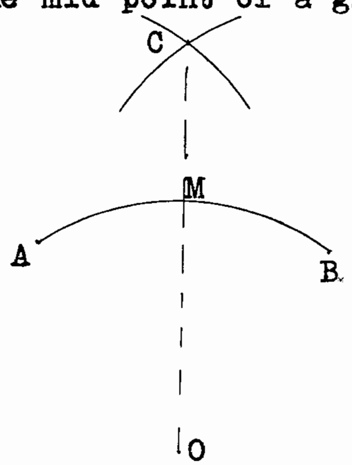
A	P O B(P)		$C_1 + L_2$
B			

C	A B	B(P)	L_2

Symbol: $C_1 + 2\ L_2$; C. S. = 8; C. E. = 5; N. O. = 3.

Construction VII

To find the mid point of a given arc.



Given arc AB, to find the midpoint of arc AB.

Construction:

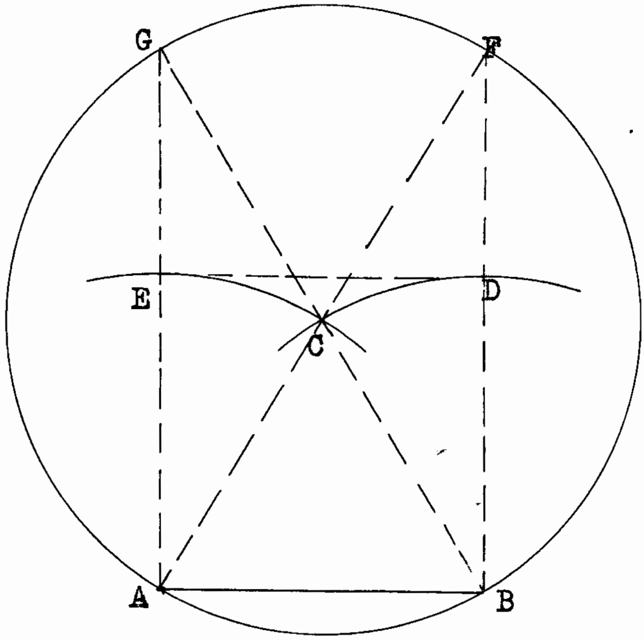
C	B(O) A(O)		$C_1 + C_2$
M	C O	O(B)	L_2

Symbol: $C_1 + C_2 + L_2$; C. S. = 8; C. E. = 5; N. O. = 3.

Proof: "The perpendicular bisector of a chord bisects the arc it subtends."

Construction VIII

To construct a square given one side.



Given: the line AB, to construct a square with each side equal to AB.

Method I.

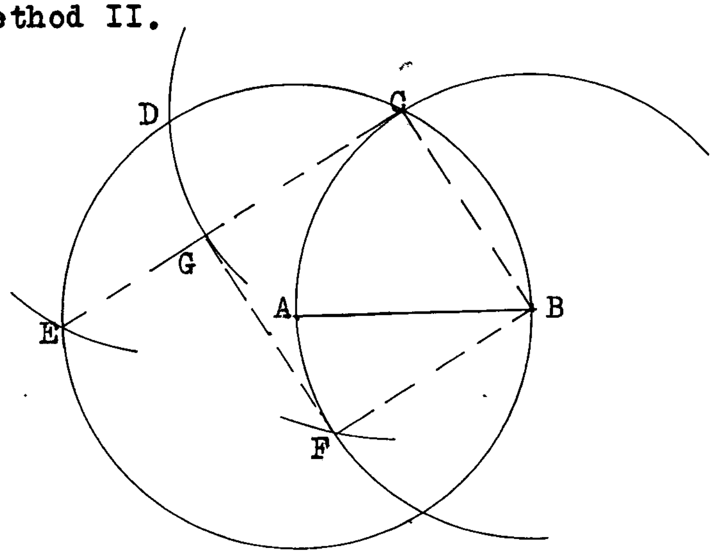
Construction:

C	B(A) A(B)		$C_1 + C_2$
D	C(B) A C		$C_1 + L_2$
E	B C	C(B)	L_2
F	F B	B(A)	L_2
G	A G	A(B)	L_2

Symbol: $2\ C_1 + C_2 + 4\ L_2$; C. S. = 19; C. E. = 11; N. O. = 8.

Proof: $\measuredangle$s EAB and DBA are rt. $\angle$s and DB = AB = EA. ($\measuredangle$ inscribed in semicircle is rt. $\angle$.)

Method II.



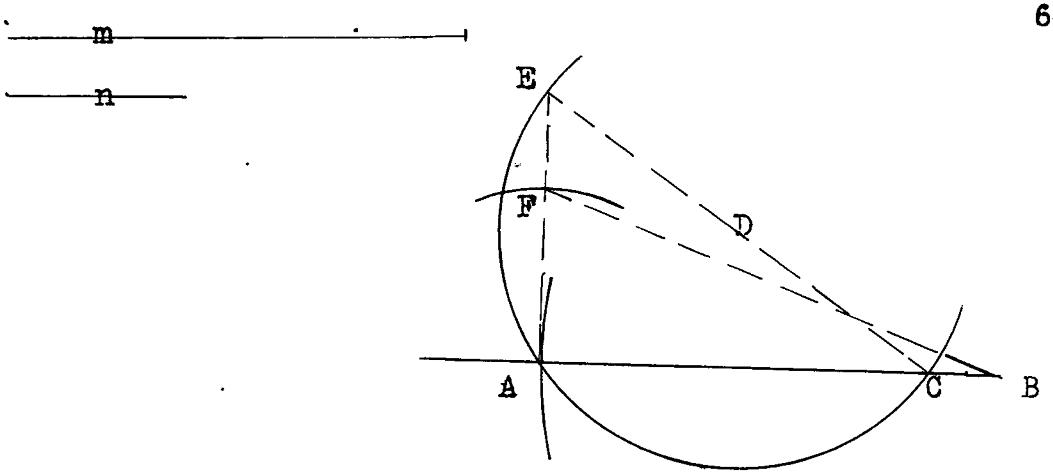
Construction:

C	A(B) B(A)		$C_1 + C_2$
D	C(AB)	A(B)	C_1
E	D(AB)	A(B)	C_1
G	C E	C(AB)	L_2
F	C(BG)	B A	C_3

Symbol: $3\ C_1 + C_2 + C_3 + L_2$; C. S. = 16; C. E. = 10; N. O. = 6.

Construction IX

To construct a right triangle having given the two sides.



Given line $\underline{m}$ and $\underline{n}$, to construct a right triangle with $\underline{m}$ and $\underline{n}$ sides.

Construction:

A	A B B(m)		$L_0 + C_3$
B			
C	D(A)	A B	C_1
D			
E	D(C	D(A)	L_2
F	E A A(n)		$L_2 + C_3$

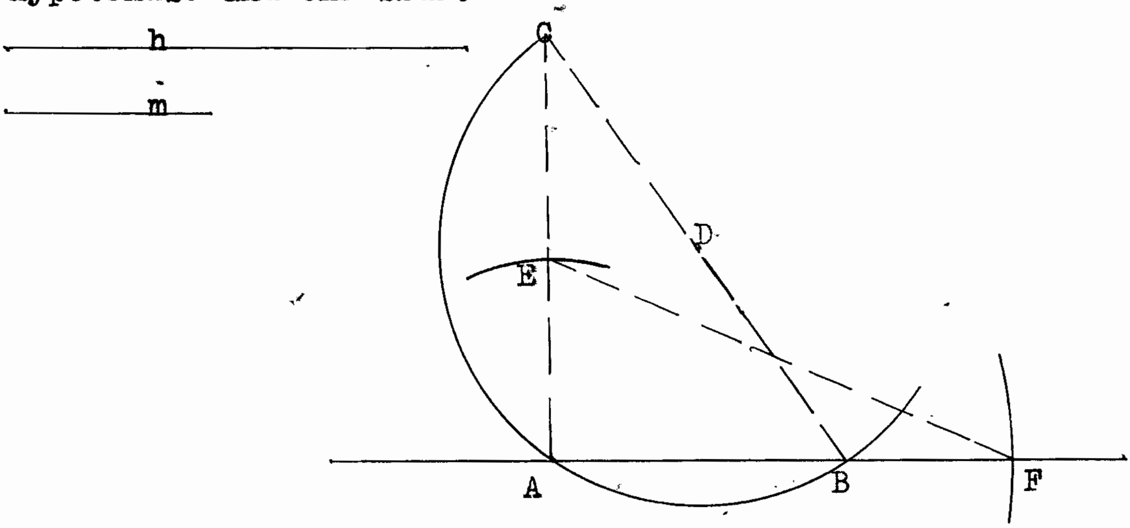
Symbol: $L_0 + 2L + C + 2C$; C. S. = 17; C. E. = 11; N. O. = 6.

<u>Corollary</u>. To construct a square equivalent to two given squares.

Construction: This will be a repetition of the above construction and the construction of VIII, Method II, which will give a C. S. = 33, C. E. = 21, and N. O. = 12.

Construction X

To construct a right triangle having given the hypotenuse and one side.



Given hypotenuse h and side m to construct a right triangle.

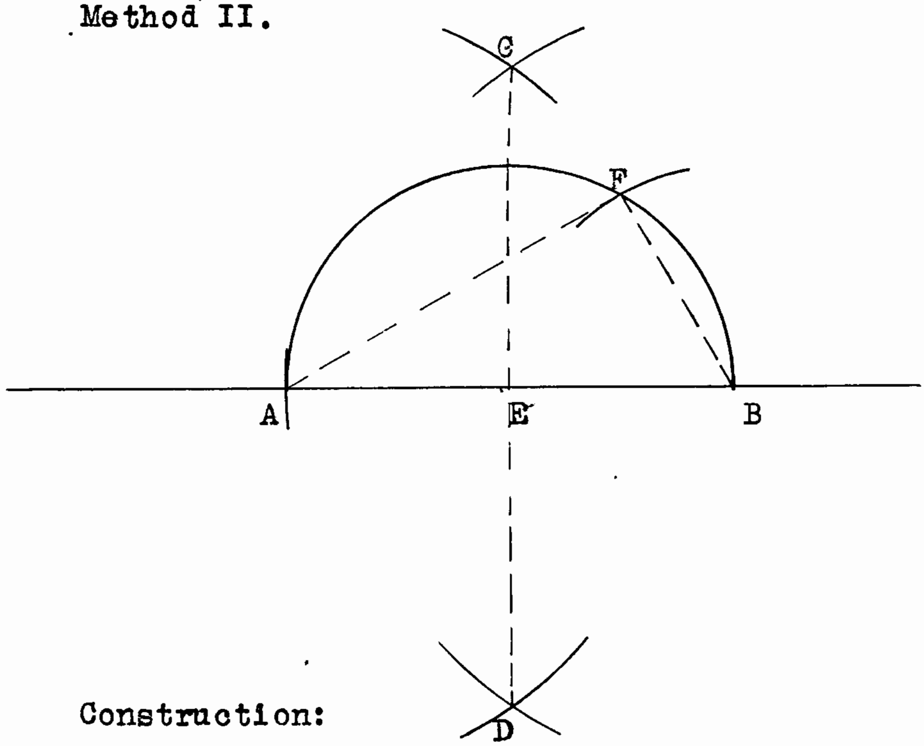
Method I.

Construction:

A	D(A) A D		$C_0 + L_0$
B		D(A) A D	
C	B D	D(A)	L_2
E	A(m) C A		$L_2 + C_3$
F	E(h)	A B	C_3

Symbol: $C_0 + L_0 + 2\ L_2 + 2\ C_3$; C. S. = 16; C. E. = 10; N. O. = 6.

Method II.



Construction:

A	B(h) A B		$L_0 + C_3$
C	B(C) A(C)		$2\ C_1$
D		B(C) A(C)	
E	C D	A B	L_2
F	E(B) B(m)		$C_2 + C_3$

Symbol: $L_0 + 2\ C_1 + C_2 + L_2 + 2\ C_3$; C. S. = 19; C. E. = 12; N. O. = 7.

Corollary: To construct a square equivalent to the difference of two given squares.

Construction: This construction may be done by the above construction and construction VIII Method II, which will give a C. S. = 32, C. E. = 20, and N. O. = 12.

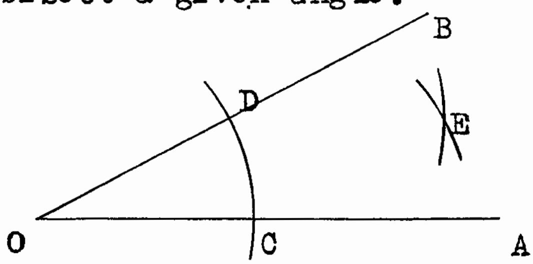
Construction XI

To find the points of intersection of a circle with a line.

This can be done by one operation (L_2) and we get C. S. = 3, C. E. = 2, and N. O. = 1.

Construction XII

To bisect a given angle.



Given angle AOB, to find the bisector.

C	O(C)	O A	C_1
D		O(C) O B	
E	C(E) D(E)		2 C_1

Symbol: 3 C_1; C. S. = 6; C. E. = 3; N. O. = 3.

Construction XIII

From a given point in a given line to draw a line making an angle equal to a given angle.

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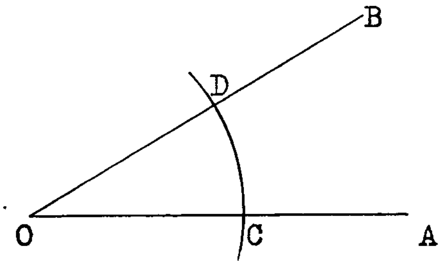
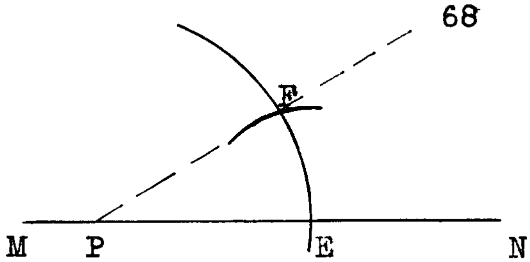




Given angle AOB and P on line MN, to draw a line from P making an angle with MN equal to AOB.

Construction:

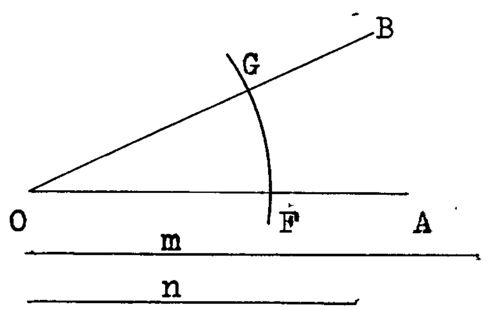
C	O(C)	O A	C_1
D		O(C) O B	
E	P(OC)	M N	C_1
F	E(CD)	P(OC)	C_3

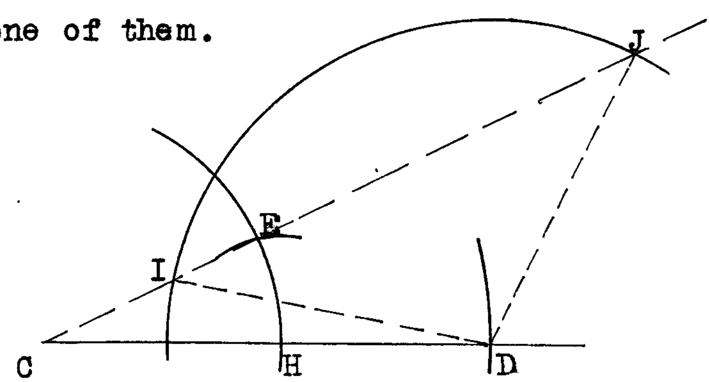
Symbol: $2\ C_1 + C_3$; C. S. = 8; C. E. = 5; N. O. = 3.

<u>Corollary 1</u>. To construct a triangle given two sides and included angle.

Construction: This requires construction XIII plus the operations $L_0 + 2\ C_3$ which gives for the corollary symbol: $L_0 + 2\ C_1 + 3\ C_3$, C. S. = 17, C. E. = 11, N. O. = 6.

<u>Corollary 2</u>. To construct a triangle given two sides and an angle opposite one of them.





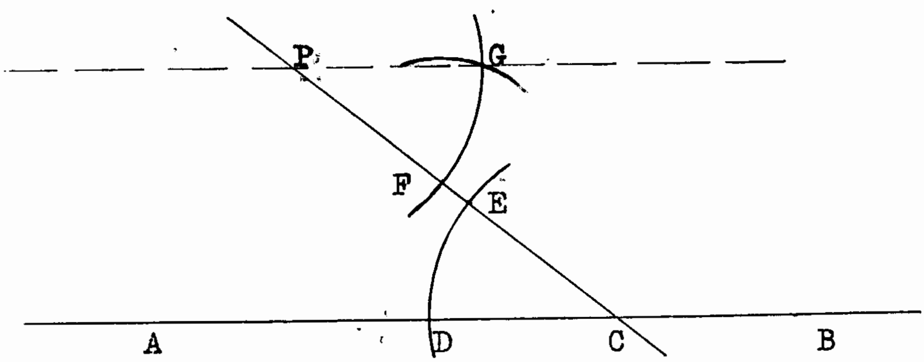
Construction:

D	C(m) C D		$L_0 + C_3$
E	H(FG) C(OF)		$C_1 + C_3$
F	O(F)	O A	C_1
G		O(F) O B	
H	C(OF)	C D	C_1
I	D(n) C E		$L_2 + C_3$
J		D(n) C E	

Symbol: $L_0 + L_2 + 3\ C_1 + 3\ C_3$; C. S. = 22; C. E. = 14; N. O. = 8.

Construction XV

Through a given external point to draw a line parallel to a given line.



Given the external point P and line AB, to draw a line through P parallel to AB.

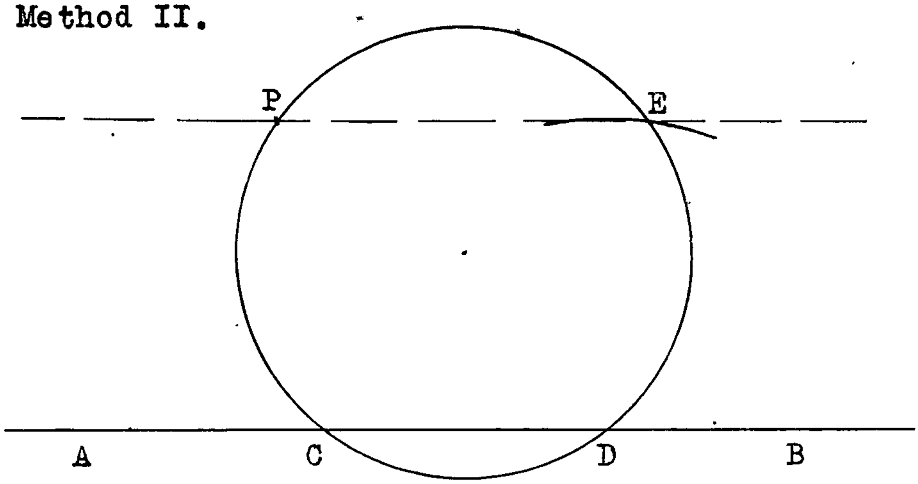
Method I.

Construction:

C	P C	A B	L_1
D	C(D)	A B	C_1
E	C(D)	P Q	
F	P(CD)	P C	C_1
G	F(DE)	P(CD)	C_3

Symbol: $L_1 + 2\ C_1 + C_3$; C. S. = 10; C. E. = 6; N. O. = 4.

Method II.



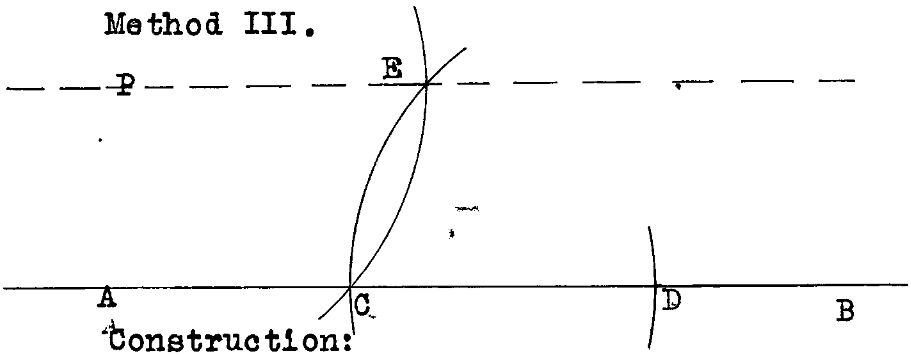
Construction:

C AND D	O(P)	A B	C_1
E	D(CP)	O P	C_3

Symbol: $C_1 + C_3$; C. S. = 6; C. E. = 4; N. O. = 2

Proof: (parallel lines intercept equal arcs on a circle.)

Method III.



Construction:

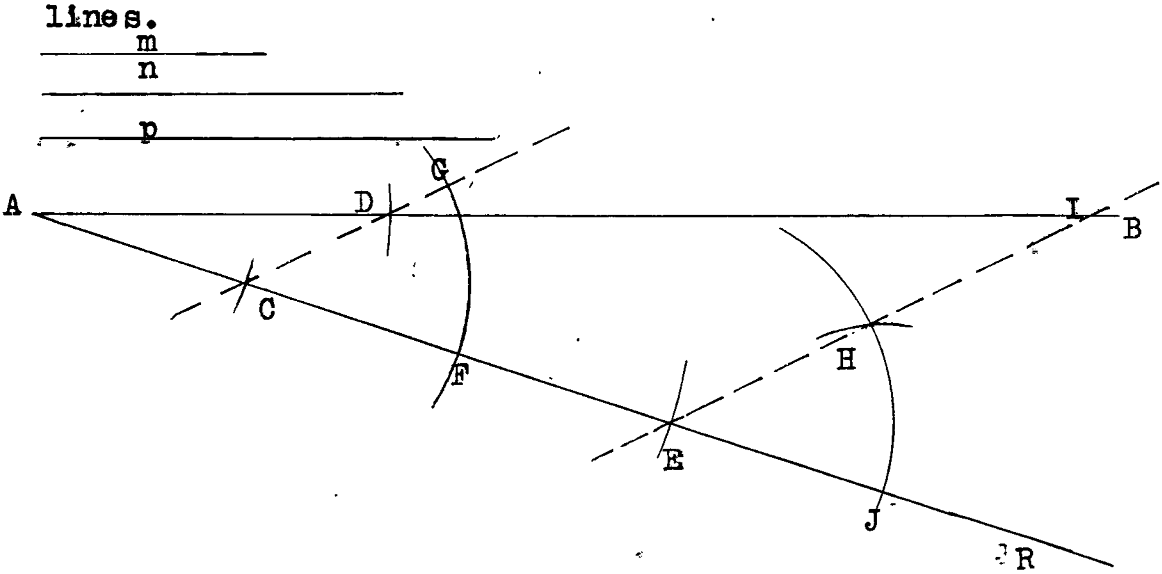
C	P(C)	A B	C_1
D	C(PC)	A B	C_1
E	D(PC)	P(C)	C_1

Symbol: 3 C_1; C. S. = 6; C. E. = 3; N. O. = 3.

Proof: Opposite sides of a rhombus are parallel.

Construction XVI

To find the fourth proportional to three given lines.



Given the three line segments m, n, and p, to find the fourth proportional.

Construction:

C	A(m) A R		$L_1 \nmid E_3$
D	A(n)	A B	C_3
E	C(p)	A R	C_3
F	C(F)	A R	E_1
G	C D	C(F)	L_2
J	E(J)	A R	C_1
H	J(FG)	E(J)	C_3
I	E H	A B	L_2

Symbol: $L_1 + 2\ C_1 + 2\ L_2 + 4\ C_3$; C. S. = 28; C. E. = 19; N. O. = 9.

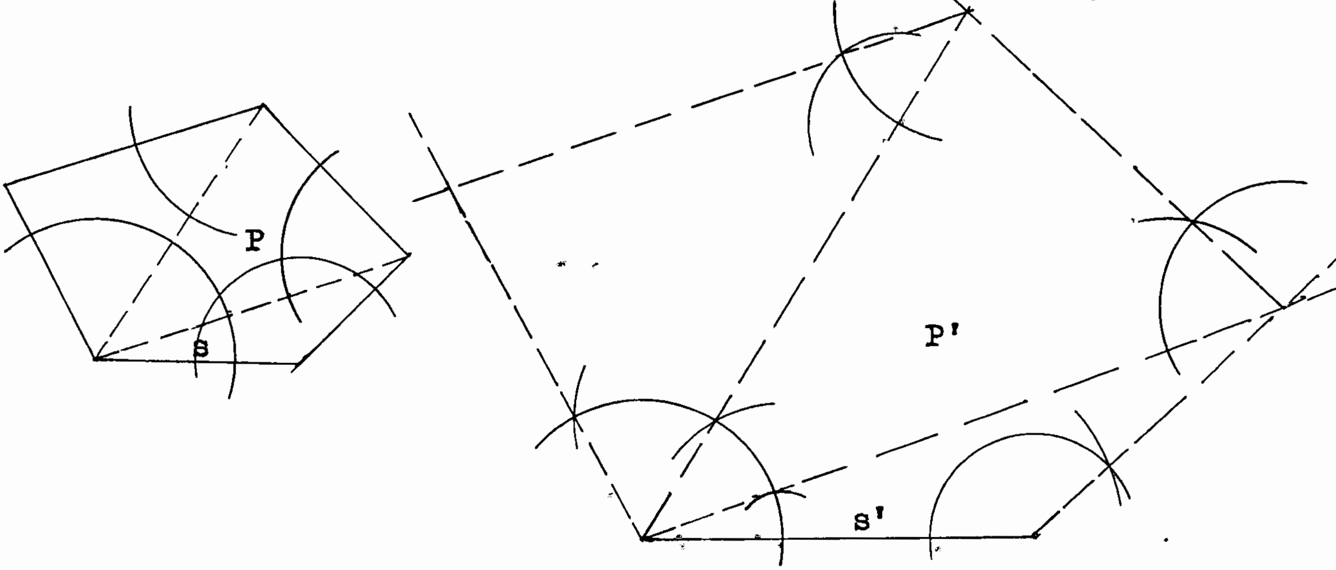
Proof: Parallel lines intercept proportional intercepts on two or more tranversals.

Corollary. To find the third proportional to two given lines.

Construction: This is a reproduction of the above construction only we use m for p and we will obtain the same measures as above.

Construction XVII

To construct a polygon similar to a given polygon.

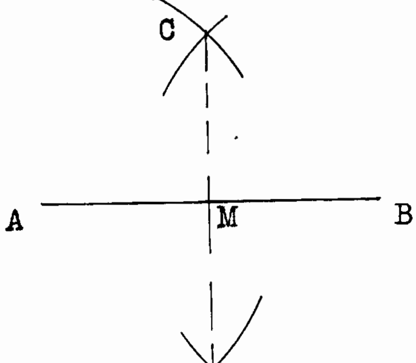


Given polygon P and s' corresponding to s, to construct a polygon similar to P.

Construction: In this construction we repeat Construction XIII six times and we draw a straight line through two given points eight times which gives us a symbol = $8\ L_2 + 8\ C_1 + 6\ C_3$, C. S. = 64, C. E. = 42, N. O. = 22.

Construction XVIII

To find a line segment equal to one half of a given line segment.



Given line segment AB, to find a line segment equal one half of AB.

Construction:

C, D	A(C) B(C)		2 C_1
M	C D	A B	L_2

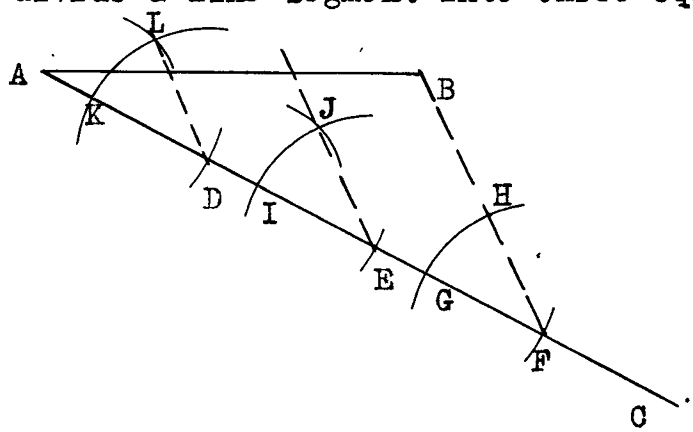
Symbol: 2 C_1 + L_2; C. S. = 7; C. E. = 4; N. O. = 3.

Corollary: To find the midpoint of a line segment.

Construction: same as construction above.

Construction XIX

To divide a line segment into three equal parts.



Given line segment AB, to divide it into three equal parts.

Construction: We construct an angle equal to a given angle twice and make the operation $L_1 + L_2$. This gives us a symbol: 7 C_1 + C_3 + L_1 + L_2, C. S. = 23, C. E. = 13, and N. O. = 10.

Construction XX

To find the point of intersection of two given lines.

Construction: Symbol = 2 L_2, C. S. = 6, C. E. = 4, and N. O. = 2.

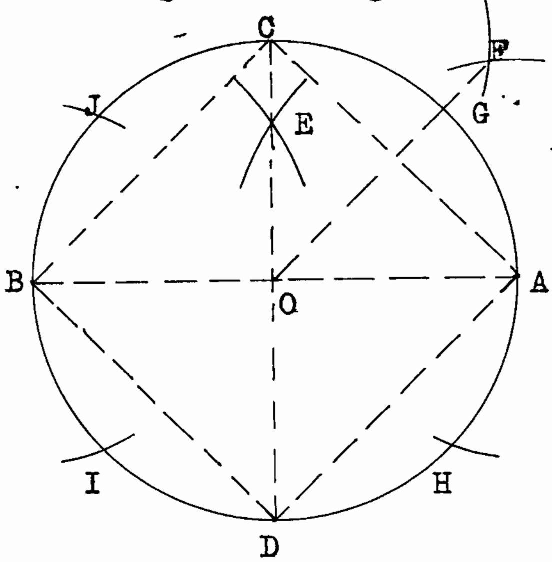
Corollary Construction XXI

To construct a line equal to the square root of two.

Construction: C F of Method II Construction VIII is the required line and it may be obtained by a reproduction of Method II. C. S. = 12, C. E. = 7, N. O. = 5.

Construction XXII

To inscribe a square in a given circle.



Given the circle with center O, to inscribe a square.

Construction:

A, B	O A	O(A)	L_1
E	B(E) A(E)		2 C_1

C, D	E O	O(A)	L_2

Symbol: $L_1 + L_2 + 2\ C_1$; C. S. = 9; C. E. = 5; N. O. = 4.

Corollary. To inscribe a regular octagon in a circle.

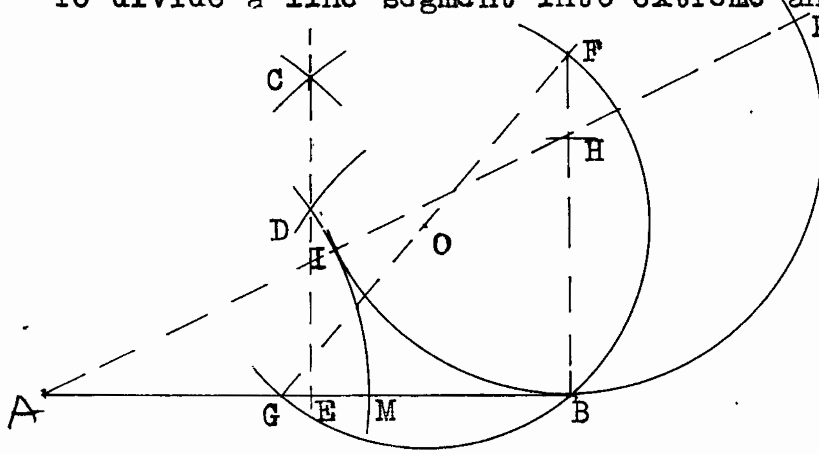
Construction: The above construction plus the following:

F	A(F) C(F)		$2\ C_1$
G	O F	O(A)	L_2
H	A(G)	O(A)	C_2
I, J	B(AG)	O(A)	C_1

Symbol: $5\ C_1 + C_2 + L_1 + 2\ L_2$; C. S. = 21; C. E. = 12; N. O. = 9.

Construction XXIII

To divide a line segment into extreme and mean ratio.



Given the line segment AB, to divide AB into extreme and mean ratio.

Construction:

C, D	B(D) A(D)		2 C_1
E	C D	A B	L_2
F	O(B) G O		$C_1 + L_2$
G		O(B) A B	
H	B(E) F B		$C_2 + L_2$
I	H(B) A H		$C_1 + L_2$
M	A(I)	A B	C_2

Symbol: 4 C_1 + 2 C_2 + 4 L_2; C. S. = 26; C. E. = 16; N. O. = 10.

Proof: AK : AB = AB : AI

AK - AB : AB = AB - AI : AI

or AM : AB = MB : AM

or AB : AM = AM : MB

Construction XXIV

To inscribe a regular decagon in a circle.

Construction: Suppose AB of Construction XXIII is the radius of the circle in which the regular decagon is to be inscribed.

We first divide AB into extreme and mean ratio, then with radius AB we describe arcs on the circle and we obtain ten equal arcs.

Symbol: $12\ C_1 + 2\ C_2 + 4\ L_2 + C_3$; C. S. = 46; C. E. = 27; N. O. = 19.

CHAPTER V

CONSTRUCTION WITH COMPASSES, RULER, AND TRIANGLES

In this chapter we will do the same constructions of Chapter III and we will make use of the triangles to simplify our constructions. The constructions, however, which can not be simplified by use of the triangles we will not repeat. As we all know, triangles can be used to simplify constructions in which perpendiculars are drawn and in which the parallel postulate is constructed.

Construction I

To let fall a perpendicular from an external point upon a given line.

Construction: with the triangles this can be done in one operation, which is represented by the symbol T_3. C. S. = 4, C. E. = 3.

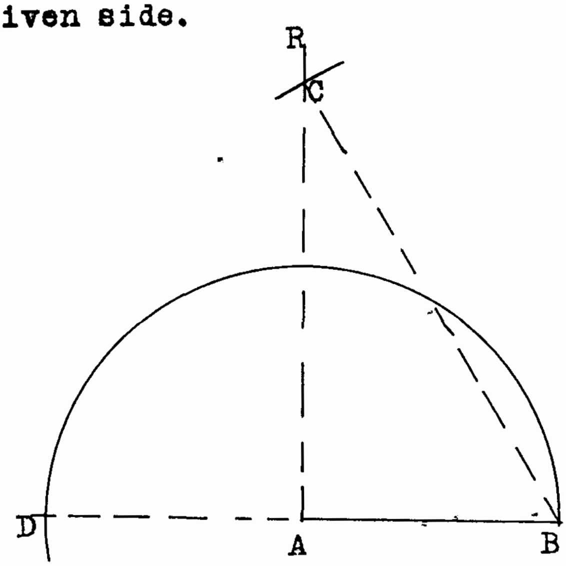
Construction II

At a given point in a given line to erect a perpendicular to the line.

Construction: with the triangles this can be done in one operation, which is represented by the symbol T_2. C. S. = 3, C. E. = 2.

Construction IV

To construct a right triangle with hypotenuse twice a given side.



Given line AB, to construct a right triangle with hypotenuse twice the side AB.

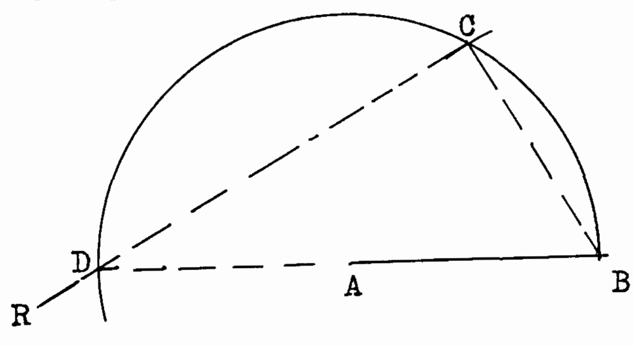
Method I.

Construction:

D	A(B) B A		$C_2 + L_2$
C	B(D) A R		$C_2 + T_2$

Symbol: $2C_2 + L_2 + T_2$; C. S. = 12; C. E. = 8; N. O. = 4.

Method II.



Construction:

C	A(B) B(A)		$C_1 + C_2$
D	C R	A(B)	T_2

Symbol: $C_1 + C_2 + T_2$; C. S. = 8; C. E. = 5; N. O. = 3.

Construction VI

To draw a tangent to a circle at a given point on the circle.

Construction: This can be done in one operation with the triangle. Erect a perpendicular at the end of the radius on the circumference. Symbol: T_2, C. S. = 3, C. E. = 2, N. O. = 1.

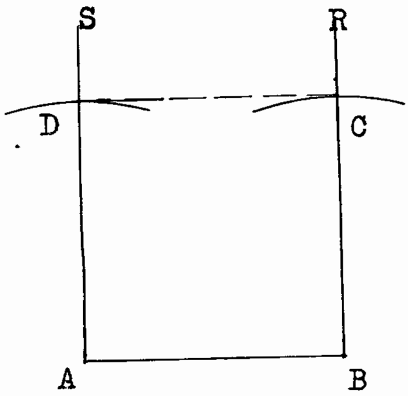
Construction VII

To find the midpoint of a given arc.

Construction: This can be done in one operation, to drop a perpendicular from a given external point to a given line and produce it. Symbol: T_3, C. S. = 4, C. E. = 3.

Construction VIII

To construct a square given one side.



Given the side AB, to construct a square.

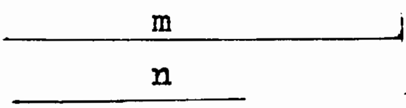
Construction:

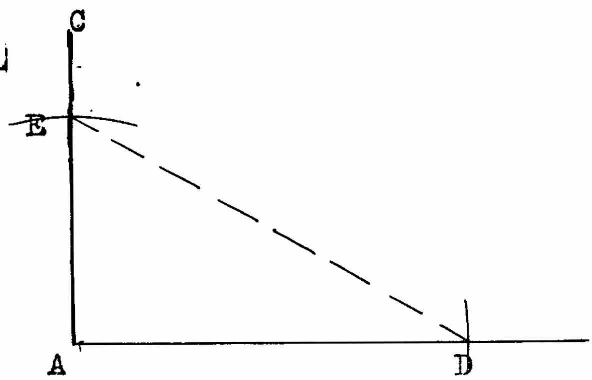
C	B(A) B R		$C_2 + T_2$
D	A(BA) A S		$C_1 + T_2$

Symbol: $C_1 + C_2 + 2T_2$; C. S. = 11; C. E. = 7; N. O. = 4.

Construction IX

To construct a right triangle having given the two sides





Given the two sides m and n, to construct a right triangle with m and n sides.

Construction:

D	A(m) A B		$C_3 + L_0$
E	A(n) A C		$C_3 + L_0$

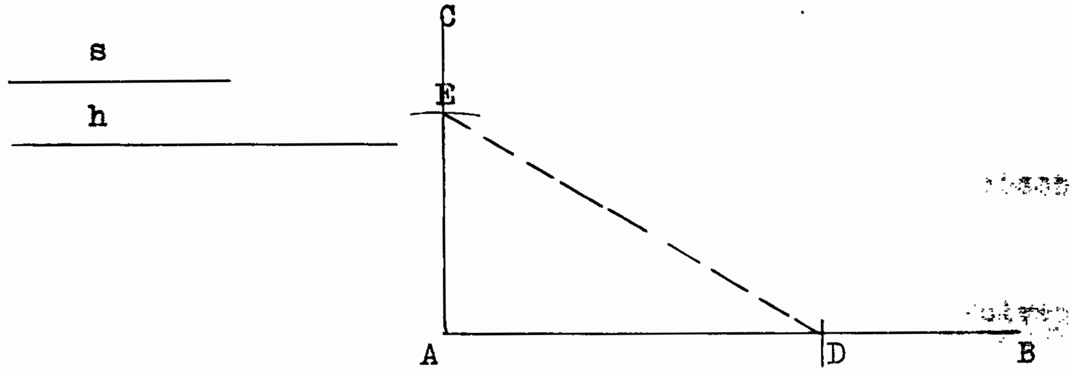
Symbol: $2L_0 + 2C_3$; C. S. = 10; C. E. = 6; N. O. = 4.

__Corollary.__ To construct a square equivalent to two given squares.

Construction: This will be the equivalent of constructions VIII and IX. C. S. = 21, C. E. = 13, and N. O. = 8.

Construction X

To construct a right triangle having given the hypotenuse and one side.



Given the line segments h and s, to construct a right triangle with h and s the hypotenuse and one side respectively.

Construction:

E	A(s) A C		$C_3 + L_0$
D	E(h) A B		$C_3 + L_0$

Symbol: $2L_0 + 2C_3$; C. S. = 10; C. E. = 6; N. O. = 4.

Corollary. To construct a square equivalent to the difference of two given squares.

Construction: This can be done by the application of constructions VIII and X. C. S. = 21, C. E. = 13, N. O. = 8.

Construction XIII--Corollary 1

To construct a triangle having given two sides and included angle.

Construction: This can be done by four operations: $T_2 + T_3 + 2C_3$. C. S. = 15, C. E. = 11.

Construction XIII--Corollary 2

To construct a triangle having given two sides and an angle opposite one of them.

Construction: This construction also involves four operations: $T_2 + T_3 + 2C_3$. C. S. = 15. C. E. = 11.

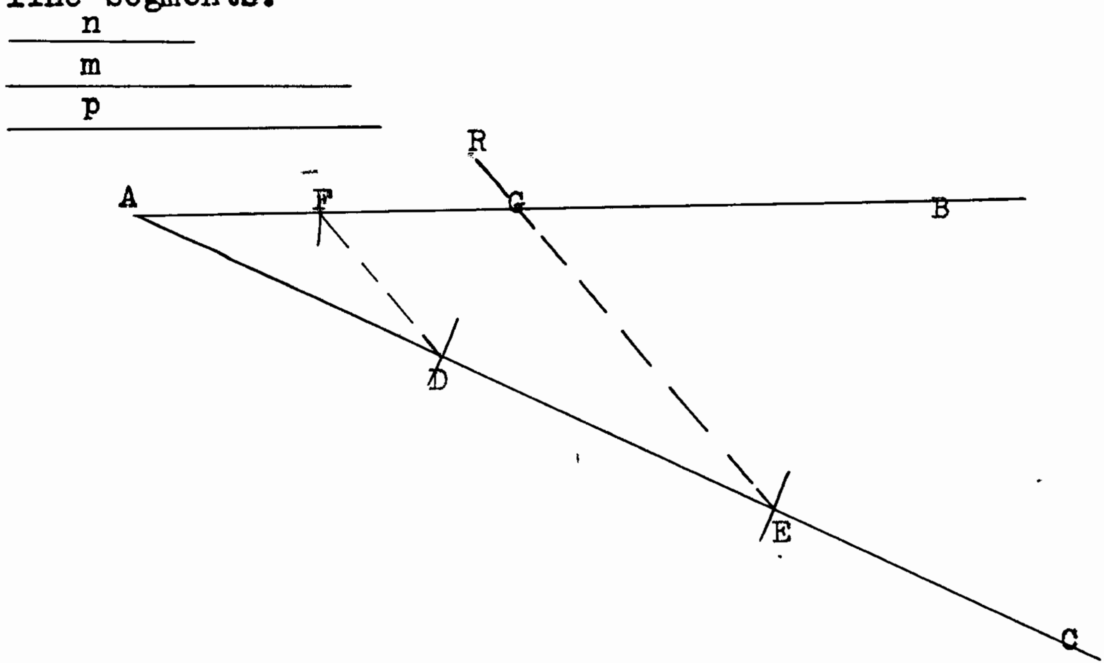
Construction XV

Through a given external point to draw a line parallel to a given line.

Construction: This construction is made by a single operation with triangles: T_3. C. S. = 4, C. E. = 3.

Construction XVI

To find the fourth proportional to three given line segments.



Given the three line segments <u>m</u>, <u>n</u>, and <u>p</u>, to find the fourth proportional.

Construction:

D	A(m) A C		$C_1 + C_3$
E	D(p)	A C	C_3
F	A(n) A B		$L_0 + C_3$
G	E R	A B	T_3

Symbol: $L_0 + L_1 + 2C_3 + T_3$; C. S. = 15; C. E. =

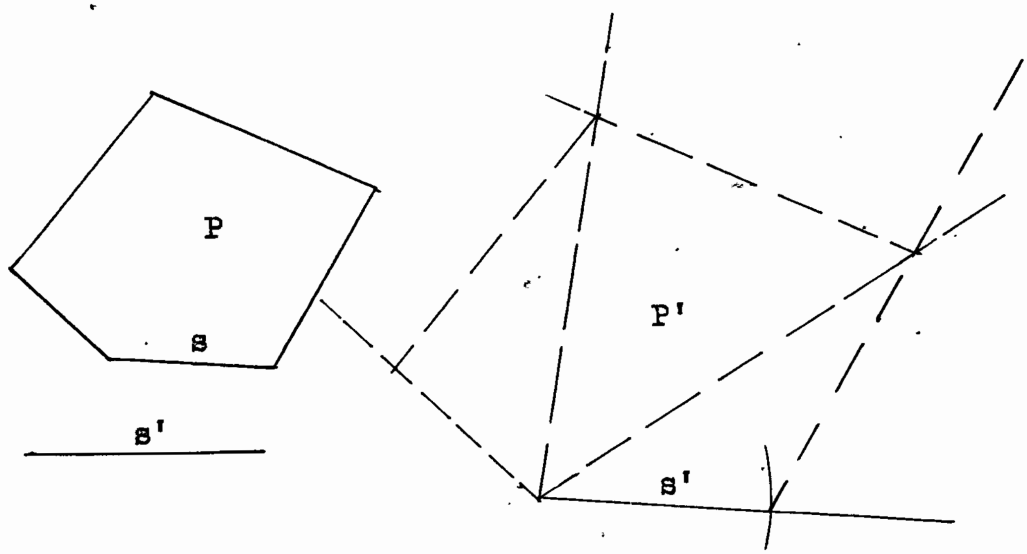
10; N. O. = 5.

Corollary. To find the third proportional to two given line segments.

Construction: This will be a reproduction of the above construction, except we use n a second time in the place of p.

Construction XVII

To construct a polygon similar to a given polygon with a given line segment as a corresponding side.

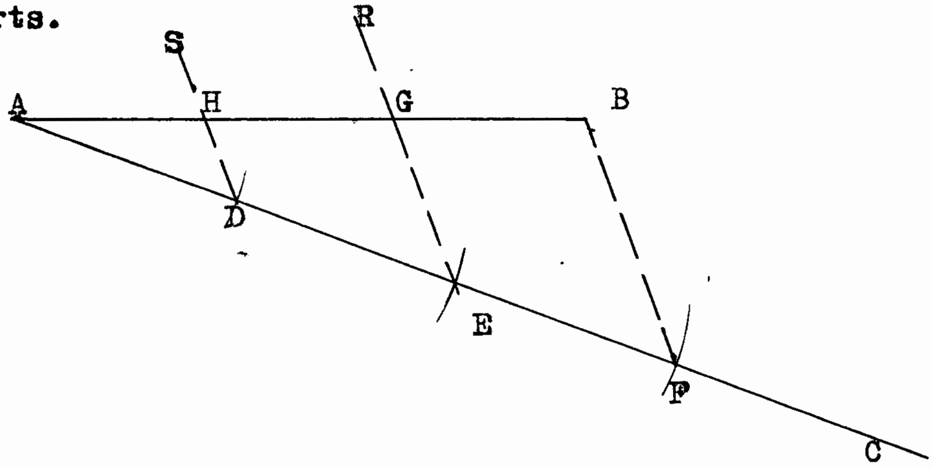


Given the polygon P and the line segment S' corresponding to the side S, to construct a polygon similar to P with S' a side corresponding to S.

Construction: In this construction we used the operation T_3 seven times and C_3 one time. Symbol: $C_3 + 7T_3$, C. S. = 32, C.E. = 24, N. O. = 8.

Construction XIX

To divide a given line segment into three equal parts.



Given the line segment AB, to divide AB in three equal parts.

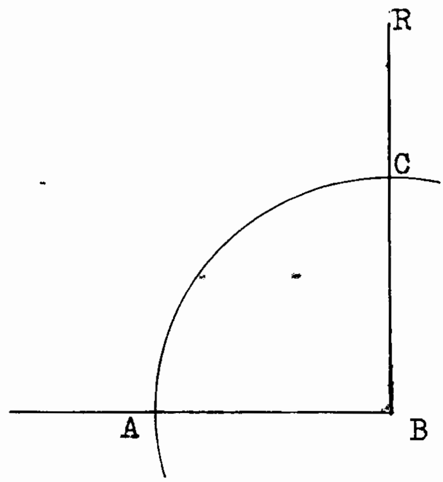
Construction:

D	A(D) A C		$L_1 + C_1$
E	D(AD)	A C	C_1
F	E(AD)	A C	C_1
G	E R	A B	T_3
H	D S	A B	T_3

Symbol: $L_1 + 3C_1 + 2T_3$; C. S. = 16; C. E. = 10; N. O. = 6.

Construction XXI--Corollary

To construct a line equal to the square root of two.



Given AB equal to one unit, to find a line segment equal to the square root of two.

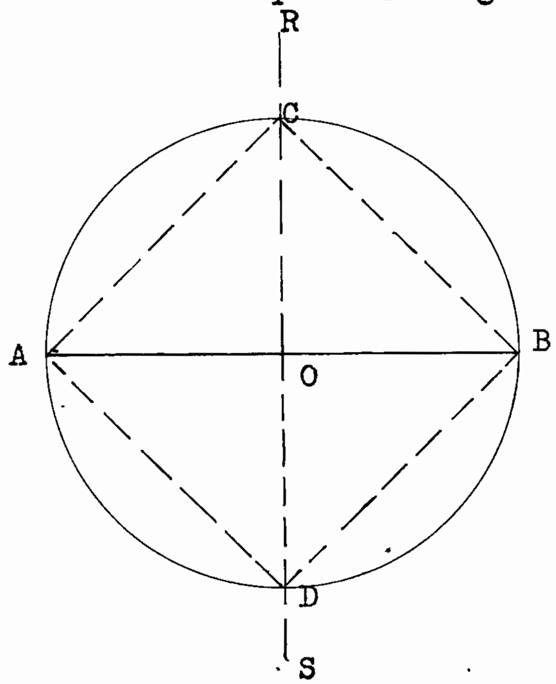
Construction:

C	B(A)	B R	$C_2 + T_2$

Symbol: $C_2 + T_2$; C. S. = 6; C. E. = 4; N. O. = 2.

Construction XXII

To inscribe a square in a given circle.



Given the circle with center O, to inscribe a square.

Construction:

A, B	A O	O(A)	L_1
C	O R	O(A)	T_3
D	O S	O A	

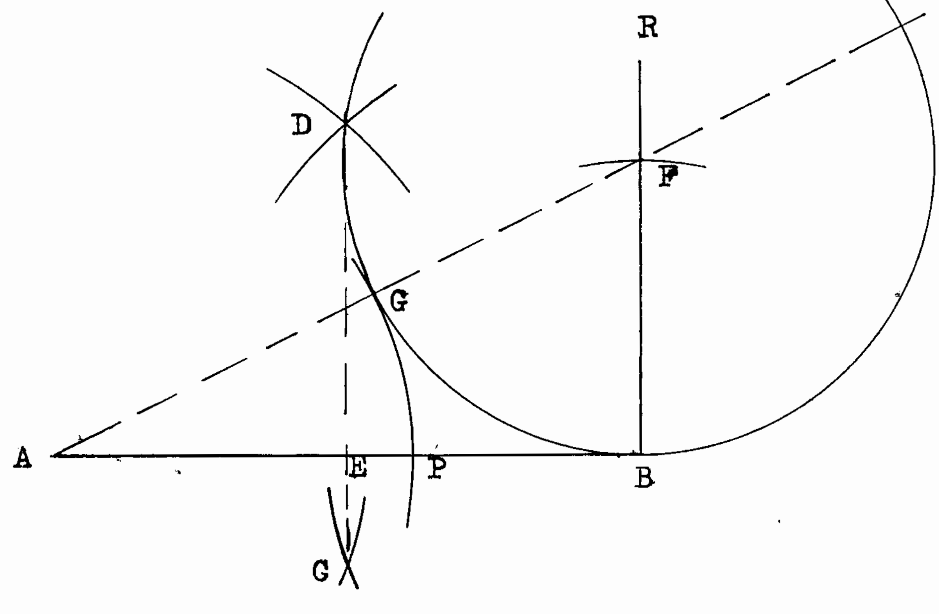
Symbol: $L_1 + T_2$; C. S. = 5; C. E. = 3; N. O. = 2.

<u>Corollary</u>. To inscribe a regular octogon in a circle.

Construction: This would be the same as the construction above plus two T_3. C. S. = 13, C. E. = 9, N. O. = 4.

<u>Construction XXIII</u>

To divide a given line segment into extreme and mean ratio.



Given the line segment AB, to divide it into extreme and mean ratio.

Construction:

D; C	A(D) B(D)		$2C_1$
E	D C	A B	L_2
F	BR, B(E)		$C_2 + T_2$
G	AF, F(B)		$C_1 + L_2$
P	A(G)	A B	C_2

Symbol: $3C_1 + 2C_2 + 2L_2 + T_2$; C. S. = 21; C. E. = 13; N. O. = 8.

Construction XXIV

To inscribe a regular decagon in a given circle.

Construction: This construction will be the same as Construction XXIII plus $C_3 + 8C_1$. Symbol: $11C_1 + 2C_2 + C_3 + 2L_2 + T_2$. C. S. = 41, C. E. = 24, and N. O. = 17.

CHAPTER VI

CONCLUSION

Constructions	Compasses			Compasses and Ruler			Compasses, Ruler, and Triangles		
	C.S.	C.E.	N.O.	C.S.	C.E.	N.O.	C.S.	C.E.	N.O.
I	6	4	2	6	4	2	4	3	1
II	9	5	4	9	5	4	3	2	1
II Cor.	9	5	4	6	4	2	6	4	2
III	4	2	2	4	2	2	4	2	2
IV	9	5	4	8	5	3	8	5	3
V	6	4	2	5	3	2	5	3	2
VI	7	4	3	8	5	3	3	2	1
VII	18	12	6	8	5	3	4	3	1
VIII	19	11	8	16	10	6	11	7	4
IX	33	21	12	17	11	6	10	6	4
IX Cor.	52	32	20	33	21	12	21	13	8
X	18	11	7	16	10	6	10	6	4
X Cor.	37	22	15	32	20	12	21	13	8
XI	20	13	7	3	2	1	3	2	1
XII	27	17	10	6	3	3	6	3	3
XIII	27	19	8	8	5	3	4	3	1
XIII Cor.1	54	37	17	17	11	6	15	11	4
XIII Cor.2	34	24	10	22	14	8	15	11	4

	C.S.	C.E.	N.O.	C.S.	C.E.	N.O.	C.S	C.E.	N.O.
XIV	11	8	3	11	8	3	11	8	3
XV	8	6	2	6	3	3	4	3	1
XVI	13	9	4	28	19	9	15	10	5
XVI Cor.	13	8	5	28	19	9	15	10	5
XVII	67	47	20	64	42	22	32	24	8
XVIII	19	12	7	7	4	3	7	4	3
XVIII Cor.	17	10	7	7	4	3	7	4	3
XIX	29	16	13	23	13	10	16	10	6
XX	34	22	12	6	4	2	6	4	2
XXI	11	6	5	11	6	5	11	6	5
XXI Cor.	15	8	7	12	7	5	6	4	2
XXII	15	9	6	9	5	4	5	3	2
XXII Cor.	23	15	8	21	12	9	13	9	4
XXIII	21	12	9	26	16	10	21	13	8
XXIV	39	22	17	46	27	19	41	24	17
Total	724	458	266	528	329	200	363	235	128

Table I.

The results obtained from the three methods of construction are summarized in Table I. Where there were two or more methods of constructing a problem, the construction with smallest C. S. and C. E. appears in the table. From the table we see that making the 33 constructions with compasses alone there were 458 coincidences and 266 circles drawn; while the same constructions were made with ruler and compasses with 329 coincidences and 200 circles or lines drawn. Thus, we make 129 more coincidences and draw 66 more circles when we use compasses alone to construct the 33 problems. Now for accuracy in construction, is the difference too great?

If we omit constructions VI, XII, XIII, XIII Cor. 1, and XX, we have 28 constructions remaining and to make them with compasses alone we only make 46 more coincidences and draw 27 more circles than we would by using compasses and ruler. This is about an average of one and one-half coincidences and one circle more to be drawn per construction. Certainly this increase in work would not be considered where accuracy is concerned.

There are five of the 33 constructions which are shorter by using compasses alone. Consider constructions VI, XVI, XVI Cor., XXIII, and XXIV. In making these with compasses alone there are 55 coincidences and 38 circles drawn, while with compasses and ruler, there are 86 coincidences and 50 circles and lines to be drawn. Sometimes a ruler and compasses construction is given in preference to a "compasses-alone" construction because the proof is shorter and simpler. This can not be a reason for not giving VI by compasses alone for the proof is the same by both methods. Now constructions XVI and XVI Cor. can not always be made by "compasses-alone" method as given in this work, but in most cases it can and when it can be used it is more than twice as short and the proof is as simple as ruler and compasses method. Construction XXIII is a very important construction. To the writer, it seems that the "compasses-alone" construction of this problem given in any Plane Geometry textbook would be a weight for the book. Since the proof is based upon the Pythagorean Theorem, this construction could be made to follow this theorem as well as the place it now occupies in most Plane Geometry texts.

Construction XV is another interesting construction. As it appears in Wentworth-Smith's Plane Geometry, it takes 6 coincidences and 4 lines and circles to make the construction, while in the construction as shown in this thesis with "compasses-alone", it only takes 6 coincidences and 2 circles to make the construction. The proof for the "compasses-alone" construction is even shorter.

Table I shows that in every case compasses, ruler, and triangles constructions are as short and in most cases shorter than those made with compasses and ruler. Constructions XXIII and XXIV are the only two constructions of the 33 that can be made shortest by compasses alone. If the 33 constructions made in this work can be made with 129 less coincidences and 66 less circles by introducing the ruler with the compasses, and if this can be made still with 94 less coincidences and 72 less lines and circles by introducing the triangles along with the compasses and ruler, why stop in Plane Geometry with the compasses and ruler? It seems to the writer that the student of Plane Geometry is able to use the triangles as well as the compasses and ruler, and it is his opinion that a Plane Geometry that does not introduce the triangles is leaving out something very important. That is, if we introduce the ruler with the compasses to simplify our constructions, then it would certainly be reasonable to go all the way and introduce the triangles.

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